(Tensor) Triangular Geometry

Pablo S. Ocal (pablo.ocal@oist.jp)

Okinawa Institute of Science and Technology

Motivation

Let $\mathcal{C}$ be a monoidal triangulated category. That is:

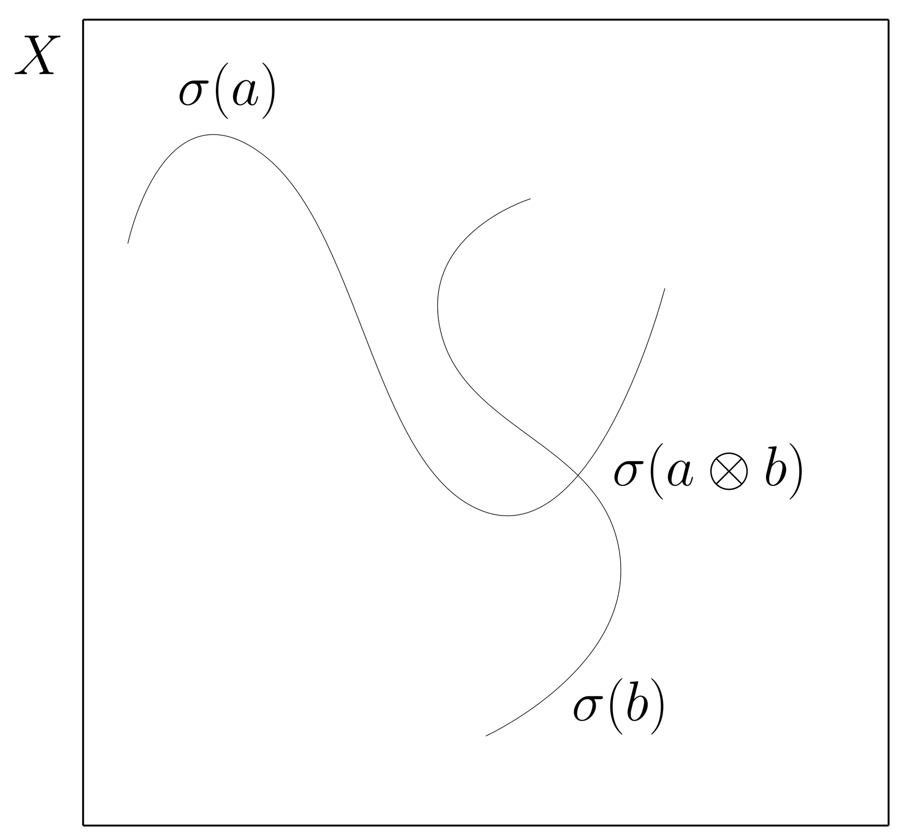
- $\mathcal{C}$ is essentially small,
- $\mathcal{C}$ is an additive category,
- $T : \mathcal{C} \to \mathcal{C}$ is an exact functor,
- there is a collection of exact triangles $a \to b \to c \to Ta$ of $\mathcal{C}$,
- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a symmetric biexact functor,
- $1 \in \mathrm{Obj}(\mathcal{C})$ is the monoidal unit.

We want to do geometry with $\mathcal{C}$. That is, we want to draw a picture that captures the essential information within $\mathcal{C}$. Mathematically this is done via a function σ which assigns to each object a of $\mathcal{C}$ a closed subset of a topological space X. Our goal is to find such a function.

$$\sigma : \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Closed}(X)$$

This function should be compatible with the structure of $\mathcal{C}$. Namely, we have the following wish list:

(SD1) $\sigma(0) = \emptyset$,
(SD2) $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$ for all $a, b \in \mathrm{Obj}(\mathcal{C})$,
(SD3) $\sigma(Ta) = \sigma(a)$ for all $a \in \mathrm{Obj}(\mathcal{C})$,
(SD4) $\sigma(a) \subseteq \sigma(b) \cup \sigma(c)$ for all exact triangles $a \to b \to c \to Ta$ of $\mathcal{C}$,
(SD5) $\sigma(1) = X$,
(SD6) $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$ for all for all $a, b \in \mathrm{Obj}(\mathcal{C})$.



As it is posed, our goal is very easy! It suffices to take $X = \{\star\}$ a single point, declaring $\sigma(0) = \emptyset$ and $\sigma(a) = \star$ for all nonzero $a \in \mathrm{Obj}(\mathcal{C})$.

The universal space admitting supports [Bal05]

We would like to have a space more interesting than just a point. Being ambitious, we ask about the best space to draw pictures. Mathematically, this will be the final space admitting a support datum for $\mathcal{C}$. Remarkably, this space exists: Its points are certain subcategories of $\mathcal{C}$.

Definition 1. *A* support datum *on a monoidal triangulated category $\mathcal{C}$ is a pair (X, σ) where X is a topological space and σ assigns to every $a \in \mathrm{Obj}(\mathcal{C})$ a closed subset $\sigma(a) \subseteq X$ satisfying (SD1), (SD2), (SD3), (SD4), (SD5), and (SD6).*

Theorem 2. *The pair $(\mathrm{Spc}(\mathcal{C}), \mathrm{supp})$ is the final support datum on $\mathcal{C}$, where $\mathrm{Spc}(\mathcal{C}) = \{\mathcal{P} \subsetneq \mathcal{C} \mid \mathcal{P}$ prime thick triangulated tensor ideal$\}$ and $\mathrm{supp}(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{C}) \mid a \notin \mathcal{P}\}$ for all $a \in \mathrm{Obj}(\mathcal{C})$.*

Being *final* means that if (X, σ) is another support datum on $\mathcal{C}$, then there exists a continuous function $f : X \to \mathrm{Spc}(\mathcal{C})$ such that $\sigma(a) = f^{-1}(\mathrm{supp}(a))$ for all $a \in \mathrm{Obj}(\mathcal{C})$. In other words, all support datum (X, σ) can be obtained from $(\mathrm{Spc}(\mathcal{C}), \mathrm{supp})$.

Theorem 3. *Let X be a quasi-compact quasi-separated scheme, then:*

$$\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \cong X.$$

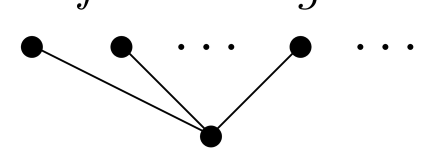
Theorem 4. *Let R be a commutative Noetherian ring, then:*

$$\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(R)) \cong \mathrm{Spec}(R).$$

Theorem 5. *Let G be a finite group, then:*

$$\mathrm{Spc}(\mathrm{stmod}(\Bbbk G)) \cong \mathrm{Proj}(\mathrm{H}^{\bullet}(G, \Bbbk)).$$

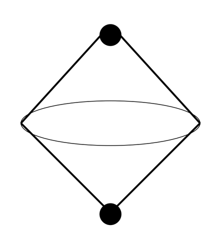
Example 6. *The Zariski spectrum of the integers.*

$\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(\mathbb{Z})) \cong$ 

Example 7. *Let $\Bbbk$ be an algebraically closed field.*

$\mathrm{Spc}(\mathrm{D}^{\mathrm{b}}(\Bbbk[x])) \cong \mathbb{A}^1_{\Bbbk} \cong$

Example 8. *Let $\Bbbk$ be a field of characteristic 2.*

$\mathrm{Spc}(\mathrm{stmod}(\Bbbk(C_2 \times C_2))) \cong$ 

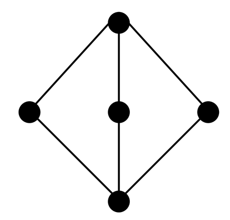
Generalizing to (non-monoidal) triangulated categories [BO24]

What happens if our category does not have a monoidal structure? We no longer need to care about the requirements involving the tensor product (SD5) and (SD6), but we can still ask for a universal space where pictures can be drawn. Surprisingly, this space is still an interesting one!

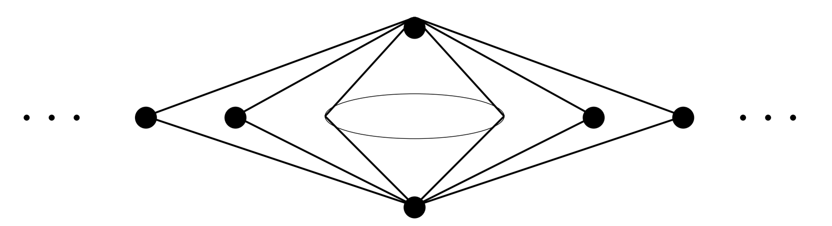
Theorem 9. *Let $\mathrm{Sp}(\mathcal{C}) = \{\mathcal{T} \subseteq \mathcal{C} \mid \mathcal{T}$ thick subcategory$\}$ and $\mathrm{sup}(a) = \{\mathcal{T} \in \mathrm{Sp}(\mathcal{C}) \mid a \notin \mathcal{T}\}$ for all $a \in \mathrm{Obj}(\mathcal{C})$. The pair $(\mathrm{Sp}(\mathcal{C}), \mathrm{sup})$ is the final support datum on $\mathcal{C}$.*

Proof. Let (X, σ) be a support datum on $\mathcal{C}$, then $f : X \to \mathrm{Sp}(\mathcal{C})$ defined by $f(x) = \{a \in \mathcal{C} \mid x \notin \sigma(a)\}$ is the desired unique continuous map. □

Example 10. *Let $\Bbbk$ be a field.*

$\mathrm{Sp}(\mathrm{D}^{\mathrm{b}}(\Bbbk A_2)) \cong$ 

Example 11. *Let $\Bbbk$ be an algebraically closed field.*

$\mathrm{Sp}(\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(\mathbb{P}^1_{\Bbbk}))) \cong$ 

References

[Bal05] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 2005.

[BO24] P. Balmer, P. S. Ocal. Universal support for triangulated categories. *C. R. Math. Acad. Sci. Paris*, 2024.

[BCR97] D. J. Benson, J. F. Carlson, J. Rickard. Thick subcategories of the stable module category. *Fund. Math.*, 1997.

[BIK97] D. J. Benson, S. B. Iyengar, H. Krause. Stratifying modular representations of finite groups. *Ann. of Math. (2)*, 2011.

[Brü07] K. Brüning. Thick subcategories of the derived category of a hereditary algebra. *Homology Homotopy Appl.*, 2007.

[KS19] H. Krause, G. Stevenson. The derived category of the projective line. *EMS Ser. Congr. Rep.*, 2019.

[Nee92] A. Neeman. The chromatic tower for $D(R)$. *Topology*, 1992.

[Tho97] R. W. Thomason. The classification of triangulated subcategories. *Compos. Math.*, 1997.