TOWARDS A RELATIVE SUPPORT THEORY

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June 21, 2022.

1

REPRESENTATIONS OF A FINITE GROUP

Strong Maschke's Theorem: kG is semisimple if and only if the characteristic of k does not divide the order of G.

Artin-Wedderburn Theorem: Semisimple rings are isomorphic to a product of finitely many matrix rings over division rings.

We can measure the failure of semisimplicity using the stable category.

2

THE BALMER SPECTRUM

Commutative algebra:		Tensor triangular geometry:
R ring	algebraic object	$\mathcal{K}$ $\otimes$-Δ-cat
$\rightsquigarrow$	$\rightsquigarrow$	$\rightsquigarrow$
$\mathrm{Spec}(R)$	topological space	$\mathrm{Spc}(\mathcal{K})$

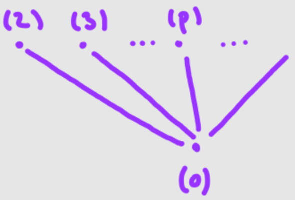
This comes with a universal notion of support that detects thick subcategories.

THIS IS USEFUL!

Example: R commutative Noetherian: $\mathrm{Spec}(R) \simeq \mathrm{Spc}(D^{perf}(R)) \simeq \mathrm{Spc}(K^b(\mathrm{proj}\, R))$.

$R = \mathbb{Z}$:

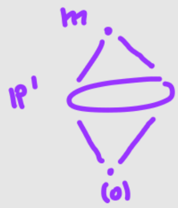
$\mathrm{Spec}(\mathbb{Z}) \simeq \mathrm{Spc}(K^b(\mathrm{proj}\, \mathbb{Z}))$



$G = \mathbb{Z}_2 \times \mathbb{Z}_2$:

$\mathrm{Spc}(\mathrm{st}(\mathrm{mod}\, kG)) \simeq \mathrm{Spc}\left(\frac{D^b(\mathrm{mod}\, kG)}{K^b(\mathrm{proj}\, kG)}\right)$

Lattice:



Max$(H^\bullet(G,k))$:

Proj$(H^\bullet(G,k))$:

SUPPORT THEORIES

Depending on the object of interest, they specialize in different homologies:

G group $\longrightarrow$ $H^\bullet(G,k)$ group cohomology

A Hopf algebra $\longrightarrow$ $H^\bullet(A,k)$ Hopf cohomology

A unital associative algebra $\longrightarrow$ $H^\bullet(A,A)$ Hochschild cohomology

If there is a natural subalgebra $B \subset A$, these theories ignore it.

RELATIVE HOCHSCHILD COHOMOLOGY

Can handle natural subalgebras: for $B \subseteq A$ unital algebras:

$HH^i(A) := \mathrm{Ext}^i(A,A)$ and $HH^\bullet(A) := \bigoplus HH^i(A)$

$HH_{(A,B)}(A) = \mathrm{Ext}_{(A^e,B^e)}(A,A)$ and $HH_{(A,B)}(A) = \bigoplus_{i\in\mathbb{N}} HH_{(A,B)}(A)$.

Theorem: 1. $HH^{\bullet}_{(A,B)}(A)$ is a graded commutative algebra with a cup product.

2. $HH^{\bullet-1}_{(A,B)}(A)$ is a graded Lie algebra.

3. $HH^{\bullet}_{(A,B)}(A)$ is a Gerstenhaber algebra.

Relative Homological Algebra ⑥

Let $B \subseteq A$ unital subring.

(A,B)-exact:

$$\cdots \longrightarrow M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \cdots$$

(i) $\ker(d_i) = \mathrm{im}(d_{i+1})$ ⟵ A-exact.

(ii) $M_i \cong \ker(d_i) \oplus Q_i$ in mod B.

Equivalently:

$$\begin{tikzcd} \cdots \arrow[r] & M_i \arrow[r, "d_i"] & M_{i-1} \arrow[l, bend left, "s_i"] \arrow[r] & \cdots \end{tikzcd}$$

(1) Over mod B we have:

$d_i d_{i+1} = 0$

$d_{i+1} s_i + s_i d_i = 1_{M_i}$

(2) Over mod B $M_\bullet$ is split exact.

Special Modules ⑦

(A,B)-free: $A \otimes_B X$, X in mod B.

$$\begin{tikzcd} \mathrm{mod}\,A \arrow[r, phantom, "\perp"] & \mathrm{mod}\,B \arrow[l, bend right, "A\otimes_B -"'] \end{tikzcd}$$

(A,B)-projective:

$$\begin{tikzcd} & & P \arrow[dll, dashed, "h'_A"'] \arrow[d, "h_A"] & \\ M \arrow[rr, "g_A"] & & N \arrow[ll, bend left, "s_B"] \arrow[r] & 0 \end{tikzcd}$$

Bottom row is (A,B)-exact.

(*) **(A,B)-flat:** For every (A,B)-exact $0 \to L \to M \to N \to 0$ then:

$$0 \longrightarrow L \otimes_A F \longrightarrow M \otimes_A F \longrightarrow N \otimes_A F \longrightarrow 0 \quad \text{is } (\mathbb{Z},\mathbb{Z})\text{-exact.}$$

Relative Long Exact Sequence: Tor ⑧

Theorem: Let $0 \to K \to L \to M \to 0$ be an (A,B)-exact sequence of right A-modules. Then for every left A-module N:

$$\cdots \rightleftarrows \mathrm{Tor}_{i+1}^{(A,B)}(M,N) \to \mathrm{Tor}_i^{(A,B)}(K,N) \rightleftarrows \mathrm{Tor}_i^{(A,B)}(L,N) \rightleftarrows \mathrm{Tor}_i^{(A,B)}(M,N) \to \cdots$$

is split exact in 2-out-of-3 terms.

Application

9

Theorem: (Relative Künneth Theorem) Let $(M_\cdot, m_\cdot)$ be a complex of right A-modules in the relative setting. Let $(N_\cdot, n_\cdot)$ be a complex of left A-modules in the relative setting. Then:

$$\bigoplus_{r+s=i} H_r(M_\cdot) \otimes_A H_s(N_\cdot) \rightleftarrows H_i(M_\cdot \otimes_A N_\cdot) \rightleftarrows \bigoplus_{r+s=i-1} \mathrm{Tor}_1^{(A,B)}(H_r(M_\cdot), H_s(N_\cdot))$$

are split short exact sequences of $\mathbb{Z}$-modules.

10

Thank you!

(A,B)-Flat

(11)

⊛ not the usual definition

For every (A,B)-exact $0 \to L \to M \to N \to 0$ then:

$$0 \to L \otimes_A F \to M \otimes_A F \to N \otimes_A F \to 0 \quad \text{is } (\mathcal{K},\mathcal{K})\text{-exact.}$$

Theorem: The following are equivalent:

(1) F is (A,B)-flat.

(2) $\mathrm{Tor}_i^{(A,B)}(M,F)=0$ for all M and i.

(3) $\mathrm{Tor}_1^{(A,B)}(M,F)=0$ for all M.

Remark:

(A,B)-flat modules preserve (A,B)-exact sequences:

$(M_\cdot, d_\cdot)$ right (A,B)-exact then

$(M_\cdot \otimes_A F, d_\cdot \otimes 1_F)$ is $(\mathcal{K},\mathcal{K})$-exact.

Application

(12)

⊛ (A,B)-flat is unusual

Given $0 \to L \rightleftarrows M \rightleftarrows N \to 0$ (A,B)-exact:

F (A,B)-flat:

$$0 \to L \otimes_A F \rightleftarrows M \otimes_A F \rightleftarrows N \otimes_A F \to 0 \quad \text{is } (\mathcal{K},\mathcal{K})\text{-exact.}$$

F "relatively flat": Weibel

$$0 \to L \otimes_A F \to M \otimes_A F \to N \otimes_A F \to 0 \quad \text{is exact.}$$

Proposition: F is (A,B)-flat $\Leftrightarrow$ F is relatively flat.

Application

(13)

Proposition: F is (A,B)-flat $\Leftrightarrow$ F is relatively flat.

Proof: Given $0 \to L \rightleftarrows M \rightleftarrows N \to 0$ (A,B)-exact:

$\Rightarrow$) Easy.

$\Leftarrow$) Tor: $\cdots \rightleftarrows \mathrm{Tor}_1^{(A,B)}(N,F) \to L \otimes_A F \underset{r}{\overset{f\otimes 1}{\rightleftarrows}} M \otimes_A F \underset{s}{\overset{g\otimes 1}{\rightleftarrows}} N \otimes_A F \to 0$

Relatively flat: $0 \to L \otimes_A F \xrightarrow{f\otimes 1} M \otimes_A F \xrightarrow{g\otimes 1} N \otimes_A F \to 0$

$(f\otimes 1)\, r\, (f\otimes 1) = f\otimes 1\,,\quad (g\otimes 1)\, s\, (g\otimes 1) = g\otimes 1 \quad \square.$