Hochschild cohomology and Gerstenhaber bracket of twisted tensor product algebras

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We are interested in Hochschild cohomology. Since we will be working with algebras over a field, in this case the Hochschild cohomology is essentially the Ext functor. The original way to compute this cohomology employed the "bar resolution", which is a bunch of copies of the algebra tensored together. We will look at both this and the slightly smaller Koszul resolution: their similarities will enable us to extend to both resolutions results that are originally proved in only one. Given two algebras, their twisted tensor product is the non-commutative analogue of the usual tensor product, via a map called twisting. Our goal is to understand the Hochschild cohomology of $A \otimes_\tau B$ in terms of the Hochschild cohomology of A and the Hochschild cohomology of B; in particular given resolutions of A and B, we work towards obtaining a resolution of $A \otimes_\tau B$.

For this, we use the notion of a resolution being compatible with the twisting, that is, a resolution satisfying the natural conditions saying that the structure of the modules, the differentials, and the twist, play well. We are able to prove that the bar resolution is compatible with the twist, and moreover we construct an isomorphism (which preserves all the desired structures) between the two intuitive resolutions of $A \otimes_\tau B$. From contracting homotopies of the bar resolution, this isomorphism enables us to construct the desired Gerstenhaber bracket on $HH^*(A \otimes_\tau B)$. All the proofs of the above statements use very specific techniques that enable us to easily book-keep all the maps we used, and we proved that we can restricting our usage to maps that descend to the Koszul resolution, effectively descending all results to it.

The tools that we use are basically fancy diagram chasing: we can reduce the compatibility conditions of the bar resolution to checking that some diagrams commute. These diagrams are mostly formed by successive applications of the twisting, and although they may look fairly innocent, after a few juggling they look something like what you can see on the screen. Here we are essentially switching the order of A and B, however, since this switch is provided with a structure inherited from the twisting, we need to be careful about what is compatible, and thus commutes, and what isn't, and thus doesn't.

While a lot of work still needs to be done, understanding this will give us insight into how complicated algebras behave, by decomposing them into simpler ones using the twisted tensor product. For example, the following quantum complete intersections and the Jordan plane arise from polynomial algebras, whose Hochschild cohomology is better understood.