Warped tensor products and noncommutative 2d TQFTs.

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Representation theory and tensor categories seminar.

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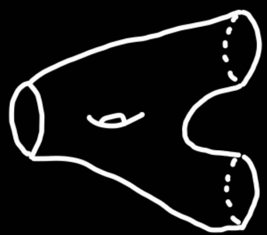
Outline

1. Topological quantum field theories.
2. Frobenius algebras.
3. Twisting and warping.
4. Applications.

1. Topological quantum field theories. [Atiyah 88]

Def: An (n+1)-cobordism between the n-manifolds M_1 and M_2 is an (n+1)-manifold N such that $\partial N = M_1 \amalg M_2$.

Example: Let $M_1 = S^1$ and $M_2 = S^1 \amalg S^1$. Then $N = \mathbb{T}^2 \setminus \{*_1, *_2, *_3\}$ is a 2-cobordism between M_1 and M_2.

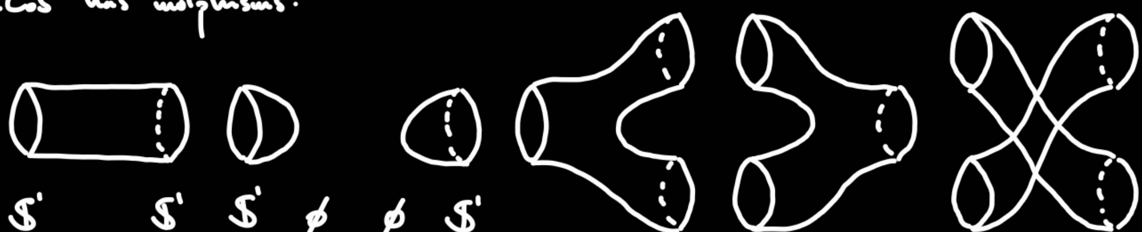


M_1 $\quad N \quad$ M_2

Def: The category nCob of n-cobordisms has for objects (n-1)-manifolds and for morphisms n-cobordisms.

Example: 2Cob has objects: $\emptyset$, S^1, and their disjoint unions.

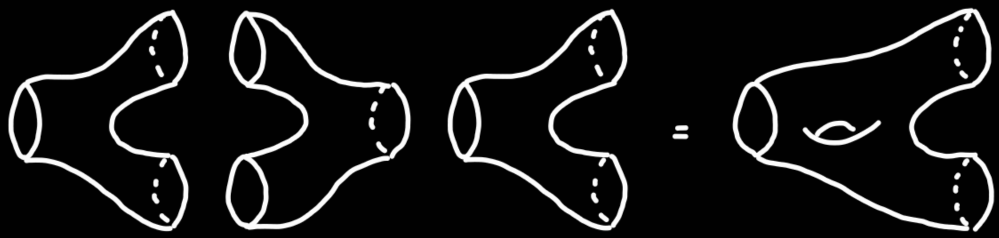
2Cob has morphisms:



$S^1 \quad S^1 \quad S^1 \quad \emptyset \quad \emptyset \quad S^1 \quad S^1 \quad S^1 \amalg S^1 \quad S^1 \amalg S^1 \quad S^1 \quad S^1 \amalg S^1 \quad S^1 \amalg S^1$

(I am sweeping under the rug that everything is given up to diffeomorphism.)

Every other morphism in 2Cob is obtained by concatenating (i.e. composing) these:



because of the classification of surfaces (compact connected orientable 2-dimensional manifold).

<u>Def:</u> An <u>n-dimensional topological quantum field theory</u> (n-d TQFT) is a symmetric monoidal functor $F : n\mathrm{Cob} \longrightarrow \mathrm{Vect}_{\Bbbk}$. Equivalently, F is a linear representation of $n\mathrm{Cob}$.

The <u>category of n-d TQFTs</u> is the category $\mathrm{SymMonFun}(n\mathrm{Cob}, \mathrm{Vect}_{\Bbbk})$ with objects n-d TQFTs and morphisms the symmetric monoidal natural transformations between them.

<u>Example:</u> $F : 2\mathrm{Cob} \longrightarrow \mathrm{Vect}_{\Bbbk}$

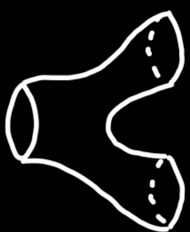
$$\emptyset \longmapsto \Bbbk$$

$$S^1 \longmapsto A = \Bbbk[x]/(x^2) \qquad \left(A = \frac{\Bbbk[x]}{(x^n)} \right)$$

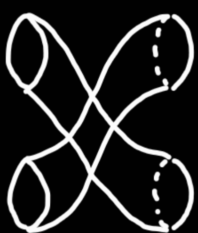
 $\longmapsto$ $id_A : A \longrightarrow A$

 $\longmapsto$ $u : \Bbbk \longrightarrow A$, $1 \longmapsto 1_A$

 $\longmapsto$ $\varepsilon : A \longrightarrow \Bbbk$, $1 \longmapsto 0$, $x \longmapsto 1$ $\qquad \left(\varepsilon : A \longrightarrow \Bbbk,\ x^i \mapsto \delta_{i,1} \right)$

 $\longmapsto$ $m : A \otimes A \longrightarrow A$

 $\longmapsto$ $\Delta : A \longrightarrow A \otimes A$, $1 \longmapsto 1 \otimes x + x \otimes 1$, $x \longmapsto x \otimes x$ $\qquad \left(\Delta : A \longrightarrow A \otimes A,\ p(x) \mapsto p(x) \otimes x + x \cdot p(x) \otimes 1 \right)$

 $\longmapsto$ $\sigma : A \otimes A \longrightarrow A \otimes A$, $a_1 \otimes a_2 \longmapsto a_2 \otimes a_1$

<u>Rmk:</u> 1. Allegedly TQFTs are related to physics, modeling space-time.

2. Instead of $\mathrm{Vect}_{\Bbbk}$, we may use $(\mathcal{C}, \otimes, a, \mathbb{1}, l, r, c)$ a symmetric monoidal category. Using Mac Lane's coherence theorem, we may regard all associators and unitors as identities, and we use the braiding $c_{A,A} : A \otimes A \xrightarrow{\sim} A \otimes A$ instead of $\sigma : A \otimes A \longrightarrow A \otimes A$.

2. Frobenius algebras. [Lam 99] [EGNO15]

Over $\mathrm{Vect}_{\Bbbk}$, there are many equivalent definitions of "Frobenius algebra". The most typical ones may be:

1. A finite dimensional algebra,
   $\varepsilon : A \longrightarrow \Bbbk$ linear map with $\mathrm{Ker}(\varepsilon)$ having <u>no</u> nontrivial left ideals.
2. A finite dimensional algebra,
   $\beta : A \otimes A \longrightarrow \Bbbk$ linear map, associative, nondegenerate.
   (a) associative:
$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A \otimes m"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes A \arrow[d, "\beta"] \\ A \otimes A \arrow[r, "\beta"] & A \end{tikzcd}$$
   (b) nondegenerate: there exists a linear map $\alpha : \Bbbk \longrightarrow A \otimes A$ such that:
$$\begin{tikzcd} A \arrow[r, "\alpha \otimes \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A \otimes \alpha"'] \arrow[dr, "\mathrm{id}_A"] & A \otimes A \otimes A \arrow[d, "\mathrm{id}_A \otimes \beta"] \\ A \otimes A \otimes A \arrow[r, "\beta \otimes \mathrm{id}_A"'] & A \end{tikzcd}$$
3. A finite dimensional algebra,
   $\phi : A \longrightarrow \mathrm{Hom}_{\Bbbk}(A, \Bbbk)$ an isomorphism of A-modules.

Since I am interested in working over a symmetric monoidal category, I want a definition that can be interpreted categorically.

<u>Def:</u> A <u>Frobenius algebra</u> in $\mathcal{C}$ is a tuple $(A, m, u, \Delta, \varepsilon)$ where A is an object in $\mathcal{C}$, $m : A \otimes A \longrightarrow A$, $u : \mathbb{1} \longrightarrow A$, $\Delta : A \longrightarrow A \otimes A$, $\varepsilon : A \longrightarrow \mathbb{1}$ are morphisms in $\mathcal{C}$, and the following diagrams commute.

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A \otimes u"] \arrow[dr, "\mathrm{id}_A"'] & A \otimes A \arrow[d, "m"] & A \arrow[l, "u \otimes \mathrm{id}_A"'] \arrow[dl, "\mathrm{id}_A"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id}_A \otimes m"] \arrow[d, "m \otimes \mathrm{id}_A"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

(A is an algebra)

$$\begin{tikzcd} A & A\otimes A \arrow[l, "id_A\otimes\varepsilon"'] \arrow[r, "\varepsilon\otimes id_A"] & A \\ & A \arrow[ul, "id_A"] \arrow[u, "\Delta"] \arrow[ur, "id_A"'] & \end{tikzcd} \qquad \begin{tikzcd} A\otimes A\otimes A & A\otimes A \arrow[l, "id_A\otimes\Delta"'] \\ A\otimes A \arrow[u, "\Delta\otimes id_A"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

(A is a coalgebra)

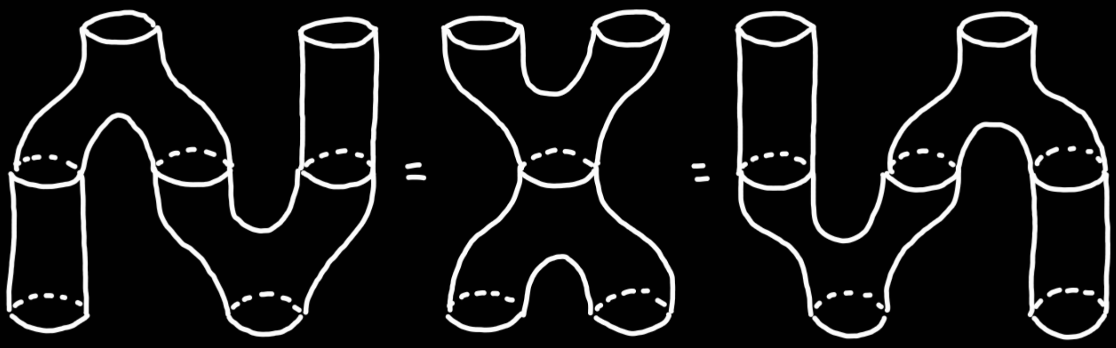
$$\begin{tikzcd} A\otimes A \arrow[rr, "id_A\otimes\Delta"] \arrow[d, "\Delta\otimes id_A"'] \arrow[dr, "m"] & & A\otimes A\otimes A \arrow[d, "m\otimes id_A"] \\ A\otimes A\otimes A \arrow[rr, "id_A\otimes m"'] & A \arrow[r, "\Delta"] & A\otimes A \end{tikzcd}$$

(Frobenius condition)

Rmk: 1. When $\mathcal{C} = \mathrm{Vect}_{\mathbb{k}}$, a Frobenius algebra is finite dimensional. In fact:

$\beta = \varepsilon m$, $\alpha = \Delta u$, and the summands of $\alpha(1)$ form a finite basis of A.

2. The Frobenius condition is a topological restriction:



3. The commutative Frobenius algebras in $\mathcal{C}$, namely the Frobenius algebras in $\mathcal{C}$ satisfying that:

$$\begin{tikzcd} A\otimes A \arrow[rr, "c_{A,A}"] \arrow[dr, "m"'] & & A\otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

commutes, form a symmetric monoidal category, denoted $\mathrm{cFrob}(\mathcal{C})$. The monoidal structure is given by:

$A\otimes B$

$m_{A\otimes B} : A\otimes B\otimes A\otimes B \xrightarrow{id_A\otimes c_{B,A}\otimes id_B} A\otimes A\otimes B\otimes B \xrightarrow{m_A\otimes m_B} A\otimes B$

$u_{A\otimes B} : \mathbb{1} \xrightarrow{u_A\otimes u_B} A\otimes B$

$\Delta_{A\otimes B} : A\otimes B \xrightarrow{\Delta_A\otimes\Delta_B} A\otimes A\otimes B\otimes B \xrightarrow{id_A\otimes c_{A,B}\otimes id_B} A\otimes B\otimes A\otimes B$

Thm: [Abrams 96][Kock 04][0. 24] There is a symmetric monoidal equivalence of categories between 2-d TQFTs and commutative Frobenius algebras given by evaluation at $\mathbb{S}^1$.

$$\mathrm{SymMonFun}(2\mathrm{Cob}, \mathcal{C}) \simeq \mathrm{cFrob}(\mathcal{C})$$

$$F \longmapsto F(\mathbb{S}^1)$$

Rmk: Frobenius algebras are important because they form one of the nicest families of self-injective algebras. Moreover, they encompass objects such as:

1. Matrix algebras.
2. More generally, by the Artin-Wedderburn theorem, <u>all</u> semisimple algebras.
3. Group algebras.
4. Algebras of characters of the representation of an algebra.
5. de Rham cohomology of connected compact oriented manifolds (by Poincaré duality).
6. More generally, any finite dimensional graded commutative algebra that is Artinian and Gorenstein.

In fact, every finite dimensional Hopf algebra is a Frobenius algebra.

Goal: Deform the tensor product of Frobenius algebras to obtain something deserving to be called a "noncommutative 2-d TQFT". These will not necessarily be formal deformations.

3. Twisting and warping.

A first naive attempt is to change the multiplication and comultiplication using twisted and cotwisted tensor products.

Def: Let A and B be algebras in $\mathcal{C}$, let $\tau: B\otimes A \to A\otimes B$ be a morphism in $\mathcal{C}$ such that the following diagrams commute.

$$\begin{tikzcd}
B\otimes A \arrow[dr, "\tau"] & A \arrow[l, "u_B\otimes id_A"'] \arrow[d, "id_A\otimes u_B"] \\
B \arrow[u, "id_B\otimes u_A"] \arrow[r, "u_A\otimes id_B"'] & A\otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B\otimes B\otimes A\otimes A \arrow[r, "id_B\otimes\tau\otimes id_A"] \arrow[d, "m_B\otimes m_A"'] & B\otimes A\otimes B\otimes A \arrow[r, "\tau\otimes\tau"] & A\otimes B\otimes A\otimes B \arrow[d, "id_A\otimes\tau\otimes id_B"] \\
B\otimes A \arrow[r, "\tau"] & A\otimes B & A\otimes A\otimes B\otimes B \arrow[l, "m_A\otimes m_B"']
\end{tikzcd}$$

(τ preserves units) $\qquad$ (τ preserves multiplications)

The <u>twisted tensor product</u> $A\otimes_\tau B$ is given by:

$A\otimes B$

$m_\tau: A\otimes B\otimes A\otimes B \xrightarrow{id_A\otimes\tau\otimes id_B} A\otimes A\otimes B\otimes B \xrightarrow{m_A\otimes m_B} A\otimes B$

$u_\tau: \mathbb{1} \xrightarrow{u_A\otimes u_B} A\otimes B$

Thm: [Čap, Schichl, Vanžura 95] The twisted tensor product $A\otimes_\tau B$ is an algebra in $\mathcal{C}$.

Proof: With shorthand.

Left unitality:

$$\begin{tikzcd}
AB \arrow[d, "u_A 11"] \arrow[dr, "11"] \arrow[dd, bend right=50, "u_\tau"'] & & & \\
AAB \arrow[r, "m_A 1"] \arrow[d, "1u_B 11"] \arrow[dr, "11u_B 1"] & AB \arrow[dr, "1u_B 1"] \arrow[drr, "11"] & & \\
ABAB \arrow[r, "1\tau 1"] \arrow[rrr, bend right, "m_\tau"'] & AABB \arrow[r, "m_A 11"] & ABB \arrow[r, "1m_B"] & AB
\end{tikzcd}$$

Right unitality follows analogously.

Associativity:

$$\begin{tikzcd}
ABABAB \arrow[r, "111\tau 1"] \arrow[rrr, bend left, "11m_\tau"] \arrow[d, "1\tau 111"] \arrow[dd, bend right=60, "m_\tau 11"'] & ABAABB \arrow[rr, "11m_A m_B"] \arrow[d, "1\tau 111"] & & ABAB \arrow[d, "1\tau 1"] \arrow[dd, bend left=60, "m_\tau"] \\
AABBAB \arrow[r, "111\tau 1"] \arrow[d, "m_A m_B 11"] & AABABB \arrow[r, "11\tau 11"] & AAABBB \arrow[r, "m_A 1 m_B 1"] \arrow[d, "m_A 1 m_B 1"] & AABB \arrow[d, "m_A m_B"] \\
ABAB \arrow[rr, "1\tau 1"] \arrow[rrr, bend right, "m_\tau"'] & & AABB \arrow[r, "m_A m_B"] & AB
\end{tikzcd}$$

$\square$.

Def: Let A and B be coalgebras in $\mathcal{C}$, let $\theta : A\otimes B \longrightarrow B\otimes A$ be a morphism in $\mathcal{C}$ such that the following diagrams commute.

$$\begin{tikzcd}
B\otimes A \arrow[r, "\varepsilon_B\otimes id_A"] \arrow[d, "id_B\otimes\varepsilon_A"'] & A \\
B & A\otimes B \arrow[l, "\varepsilon_A\otimes id_B"] \arrow[u, "id_A\otimes\varepsilon_B"'] \arrow[ul, "\theta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
B\otimes B\otimes A\otimes A & & B\otimes A\otimes B\otimes A \arrow[ll, "id_B\otimes\theta\otimes id_A"'] & A\otimes B\otimes A\otimes B \arrow[l, "\theta\otimes\theta"'] \\
B\otimes A \arrow[u, "\Delta_B\otimes\Delta_A"] & & A\otimes B \arrow[ll, "\theta"'] \arrow[r, "\Delta_A\otimes\Delta_B"] & A\otimes A\otimes B\otimes B \arrow[u, "id_A\otimes\theta\otimes id_B"']
\end{tikzcd}$$

(θ preserves counits) (θ preserves comultiplications)

The cotwisted tensor product $A\otimes^\theta B$ is given by:

$A\otimes B$

$\Delta_\theta : A\otimes B \xrightarrow{\Delta_A\otimes\Delta_B} A\otimes A\otimes B\otimes B \xrightarrow{id_A\otimes\theta\otimes id_B} A\otimes B\otimes A\otimes B$

$\varepsilon_\theta : A\otimes B \xrightarrow{\varepsilon_A\otimes\varepsilon_B} \mathbb{1}$

Thm: [Čap, Schichl, Vanžura 95] The cotwisted tensor product $A\otimes^\theta B$ is a coalgebra in $\mathcal{C}$.
[Caenepeel, Bogdan, Militaru 00]

Question: When are $A\otimes_\tau B$ and $A\otimes^\theta B$ Frobenius algebras?

Thm: [O., Oswald 24][Das, O., Plavnik 25][O., Oswald 25] The twisted and cotwisted tensor product $A \otimes_\tau^\theta B$ is a Frobenius algebra if and only if $\theta = \tau^{-1}$.

Rmk: 1. Originally Oswald and I were looking at when twisted tensor products were a Hopf algebra, which turned out to be never, for the cases we were interested in:

$$A \otimes_\tau^{\tau^{-1}} B \text{ is a Hopf algebra if and only if } \tau = c_{B,A}.$$

2. This poses a problem because the naive attempt at defining a "noncommutative TQFT" by replacing $A \otimes B$ with $A \otimes_\tau^\theta B$ cannot be specialized to commutative Frobenius algebras. It is (almost never) commutative!

3. Overall, the twist τ and the cotwist θ are too strongly related. They do not give us the flexibility we want, namely to have independent multiplication m_τ and comultiplication Δ_θ. We need to "deform" another structural map.

Def: [Das, O., Plavnik] Let $(A, m_A, u_A, \beta_A, \alpha_A)$ and $(B, m_B, u_B, \beta_B, \alpha_B)$ be Frobenius algebras in $\mathcal{C}$, let $\omega : B \otimes A \longrightarrow A \otimes B$ be a morphism in $\mathcal{C}$. The warped tensor product $A \nless_\omega B$ is given by:

$A \otimes B$,

$$m_{A\otimes B} : A\otimes B\otimes A\otimes B \xrightarrow{id_A \otimes c_{B,A} \otimes id_B} A\otimes A\otimes B\otimes B \xrightarrow{m_A \otimes m_B} A \otimes B$$

$$u_{A\otimes B} : \mathbb{1} \xrightarrow{u_A \otimes u_B} A\otimes B$$

$$\beta_\omega : A\otimes B\otimes A\otimes B \xrightarrow{id_A \otimes \omega \otimes id_B} A\otimes A\otimes B\otimes B \xrightarrow{\beta_A \otimes \beta_B} \mathbb{1}$$

$$\alpha_\omega : \mathbb{1} \xrightarrow{\alpha_A \otimes \alpha_B} A\otimes A\otimes B\otimes B \xrightarrow{id_A \otimes \omega^{-1} \otimes id_B} A\otimes B\otimes A\otimes B$$

Question: When are $A \nless_\omega B$ Frobenius algebras?

Thm: [Das, O., Plavnik] The warped tensor product $A \nless_\omega B$ is a Frobenius algebra if and only if there exist $\psi : \mathbb{1} \longrightarrow A\otimes B$ and $\varphi : \mathbb{1} \longrightarrow A \otimes B$ morphisms in $\mathcal{C}$ making the following diagrams commute.

$$\begin{tikzcd}
B\otimes A \arrow[r, "\omega"] \arrow[d, "c_{B,A}"'] & A\otimes B \\
A\otimes B \arrow[r, "id_A\otimes id_B\otimes\psi"'] & A\otimes B\otimes A\otimes B \arrow[u, "m_{A\otimes B}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "\psi\otimes\varphi"] \arrow[d, "\varphi\otimes\psi"'] \arrow[dr, "u_{A\otimes B}"] & A\otimes B\otimes A\otimes B \arrow[d, "m_{A\otimes B}"] \\
A\otimes B\otimes A\otimes B \arrow[r, "m_{A\otimes B}"] & A\otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes B \arrow[r, "id_A\otimes id_B\otimes\psi"] \arrow[d, "\psi\otimes id_A\otimes id_B"'] & A\otimes B\otimes A\otimes B \arrow[d, "m_{A\otimes B}"] \\
A\otimes B\otimes A\otimes B \arrow[r, "m_{A\otimes B}"] & A\otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes B \arrow[r, "id_A\otimes id_B\otimes\varphi"] \arrow[d, "\varphi\otimes id_A\otimes id_B"'] & A\otimes B\otimes A\otimes B \arrow[d, "m_{A\otimes B}"] \\
A\otimes B\otimes A\otimes B \arrow[r, "m_{A\otimes B}"] & A\otimes B
\end{tikzcd}$$

4. Applications.

Let's make more clear what we would like to be a "noncommutative TQFT".

Philosophically, a (commutative) TQFT should be a noncommutative TQFT.

One could then consider the candidates:

$$\begin{array}{c} \mathrm{Fun}(n\mathrm{Cob}, \mathcal{C}) \\ \cup\!\!\!\mid \\ \mathrm{MonFun}(n\mathrm{Cob}, \mathcal{C}) \\ \cup\!\!\!\mid \\ \mathrm{SymMonFun}(n\mathrm{Cob}, \mathcal{C}) \end{array}$$

but it does not feel right to ignore the symmetric nor monoidal structures. After all, they are both given by the topology encoded in $n\mathrm{Cob}$. Focusing on 2-d TQFTs, we can look at an algebraic side of things.

$$\begin{array}{ccc} \mathrm{Fun}(2\mathrm{Cob}, \mathcal{C}) & & \\ \cup\!\!\!\mid & & \\ \mathrm{MonFun}(2\mathrm{Cob}, \mathcal{C}) & & \mathrm{Frob}(\mathcal{C}) \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ \mathrm{SymMonFun}(2\mathrm{Cob}, \mathcal{C}) & \simeq & c\mathrm{Frob}(\mathcal{C}) \end{array}$$

Maybe the objects in $\mathrm{Frob}(\mathcal{C})$ deserve to be called noncommutative TQFTs. A key advantage of the algebraic side is that $c\mathrm{Frob}(\mathcal{C})$ is a full subcategory of $\mathrm{Frob}(\mathcal{C})$, and $\mathrm{Frob}(\mathcal{C})$ has a natural structure of symmetric monoidal category.

A noncommutative Frobenius algebra in $\mathcal{C}$ deserves to be called a noncommutative 2-d TQFT.

Thus the categorical data of noncommutative 2-d TQFTs could be given by a braided or symmetric monoidal structure on $\mathrm{Frob}(\mathcal{C})$ that descends suitably to $c\mathrm{Frob}(\mathcal{C})$.

Thm: [Das, O., Plavnik] Warped tensor products induce symmetric monoidal structures on $\mathrm{Frob}(\mathcal{C})$.

More precisely, for each A in $\mathrm{Frob}(\mathcal{C})$ choose $\psi_A : \mathbb{1} \longrightarrow A$ morphism in $\mathcal{C}$ such that for all (A, B) in $\mathrm{Frob}(\mathcal{C}) \times \mathrm{Frob}(\mathcal{C})$ the following diagrams commute

$$\begin{tikzcd} B \otimes A \arrow[rr, "w_{B,A}"] \arrow[d, "c_{B,A}"'] & & A \otimes B \\ A \otimes B \arrow[rr, "\mathrm{id}_A \otimes \mathrm{id}_B \otimes \psi_A \otimes \psi_B"] & & A \otimes B \otimes A \otimes B \arrow[u, "m_{c_{B,A}}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[d, "\mathrm{id}_{\mathbb{1}}"'] \arrow[dr, "\psi_{A \otimes B}"] & \\ \mathbb{1} \arrow[r, "\psi_A \otimes \psi_B"'] & A \otimes B \end{tikzcd}$$

and the morphism $w_{B,A} : B \otimes A \longrightarrow A \otimes B$ in $\mathcal{C}$ makes $A \otimes_{w_{B,A}} B$ into a Frobenius algebra.

Then:

$$\begin{aligned}\otimes : \mathrm{Frob}(\mathcal{C}) \times \mathrm{Frob}(\mathcal{C}) &\longrightarrow \mathrm{Frob}(\mathcal{C}) \\ (A, B) &\longmapsto A \otimes_{\omega_{B,A}} B\end{aligned}$$

makes $(\mathrm{Frob}(\mathcal{C}), \otimes, \mathbb{1})$ into a symmetric monoidal category.

Rmk: When $\mathcal{C} = \mathrm{Vect}_{\mathbb{k}}$, suitable choices of $\varphi_A : \mathbb{1} \longrightarrow A$ are given by invertible central elements.

Slogan: The category of 2-d noncommutative TQFTs should have $\mathrm{SymMonFun}(2\mathrm{Cob}, \mathcal{C})$ as a full subcategory, and should also have a category symmetric monoidal equivalent to $\mathrm{Frob}(\mathcal{C})$ fully embedded in it.

Maybe a good candidate is $\mathrm{Frob}(\mathcal{C})$ together with some additional categorical data that unifies the symmetric monoidal structures $\otimes$ given above.