The Gerstenhaber bracket on relative Hochschild cohomology

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Motivation

Consider A an (unital, associative) algebra over k a (unital, associative) commutative ring. Hochschild [4] introduced relative homological algebra and a relative cohomology (for algebras) with several applications:

- Cohomology theory for algebras.
- Cohomology theory for Lie algebras.

Most of its utility has been in deformation theory [3], where there is almost no need for relative homological algebra. We attempt to formalize the general constructions that are lacking detailed descriptions in the literature.

Introduction to Hochschild cohomology [6]

Consider $A^{\otimes n}$ as a bimodule over itself via $\rho_{A^{\otimes n}} = \mu_A \otimes \mathrm{id}_{A^{\otimes(n-2)}} \otimes \mu_A$:

$$\rho_{A^{\otimes n}}(a \otimes c_1 \otimes \cdots \otimes c_n \otimes b) = ac_1 \otimes \cdots \otimes c_n b.$$

Consider the sequence of A-bimodules:

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \xrightarrow{\mu_A} A \longrightarrow 0$$

where

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

This is an exact complex called the augmented bar complex.

For M an A-bimodule, consider the complex of k-modules:

$$0 \longrightarrow \mathrm{Hom}_{A-A}(A \otimes A, M) \xrightarrow{d_1^*} \mathrm{Hom}_{A-A}(A^{\otimes 3}, M) \xrightarrow{d_2^*} \cdots$$

the Hochschild cohomology $\mathrm{HH}^*(A, M)$ of A with coefficients in M is:

$$\mathrm{HH}^n(A, M) = \ker(d_{n+1}^*)/\operatorname{im}(d_n^*).$$

Moreover, there are isomorphisms of k-modules:

$$\mathrm{Hom}_{A-A}(A^{\otimes(n+2)}, M) \cong \mathrm{Hom}_k(A^{\otimes n}, M).$$

Given two cochains $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$, $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$ the cup product $f \smile g$ is defined as an element of $\mathrm{Hom}_k(A^{\otimes(m+n)}, A)$:

$$(f \smile g)(a_1 \otimes \cdots \otimes a_{m+n}) = (-1)^{mn} f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots \otimes a_{m+n})$$

and the Gerstenhaber bracket $[f, g]$ as an element of $\mathrm{Hom}_k(A^{\otimes(m+n-1)}, A)$:

$$[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f$$

where $\circ$ generalizes the composition of functions.

The structure of the Hochschild cohomology of an algebra

Theorem 1 ([2]). *The cup product induces a graded associative and graded commutative algebra structure on* $\mathrm{HH}^*(A, A)$.

Theorem 2 ([2]). *The Gerstenhaber bracket induces a graded Lie algebra structure on* $\mathrm{HH}^*(A, A)$.

In fact, the compatibility between the cup product and Gerstenhaber bracket induces a Gerstenhaber algebra structure on $\mathrm{HH}^*(A, A)$.

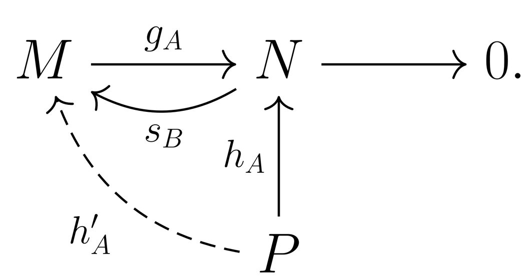
Basics of relative homological algebra

Suppose A is just a (unital, associative) ring, $1_A \in B \subseteq A$ a subring. Let:

$$\cdots \xrightarrow{t_{i+1}} C_i \xrightarrow{t_i} C_{i-1} \xrightarrow{t_{i-1}} \cdots$$

an exact sequence of A-modules, it is called (A, B) exact if $\ker(t_i)$ is a direct summand of C_i as a B-module (that is, there exists a B-homotopy).

An A-module P is said to be (A, B) projective if for every (A, B) exact sequence $M \xrightarrow{g} N \to 0$ and every A-homomorphism $h : P \to N$



there is an A-homomorphism $h' : P \to M$ with $gh' = h$.

We then have notions of:

- (A, B) projective resolution.
- Relative Comparison Theorem.
- Relative Ext groups $\mathrm{Ext}^n_{(A,B)}(M, N)$.

Remarks

For A an algebra over k a ring, we have $k \cong k \otimes k^{op} \subset A \otimes A^{op}$ and:

$$\mathrm{HH}^n(A, M) \cong \mathrm{Ext}^n_{(A \otimes A^{op}, k)}(A, M).$$

Whenever k is a field, the bar complex becomes a free resolution of A-bimodules of A, called the bar resolution. In particular we recover:

$$\mathrm{HH}^n(A, M) \cong \mathrm{Ext}^n_{A-A}(A, M).$$

Cup and bracket in relative Hochschild cohomology

Definition 3. *The relative Hochschild cohomology* $\mathrm{HH}^*(A|B, M)$ *is:*

$$\mathrm{HH}^*(A|B, M) = \mathrm{Ext}^*_{(A \otimes A^{op}, B \otimes A^{op})}(A, M).$$

Theorem 4 (O.). *The relative Hochschild cohomology* $\mathrm{HH}^*(A|B, A)$ *is a Gerstenhaber algebra.*

For a proof, we develop generalized versions of the tools Gerstenhaber [2] employed:

- relative pre-Lie systems.
- relative pre-Lie algebras.

and using a relative Künneth theorem we are able to reproduce the Yoneda product, the tensor product of complexes, and the Yoneda splice.

Tools and applications in the literature

- Construction of the Jacobi-Zariski exact sequence between Hochschild homology and cyclic (co)homology [5].
- Description of the Hochschild cohomology of a bound quiver algebra in terms of adding or deleting arrows [1].

References

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