On the Hochschild cohomology ring of some twisted tensor products of algebras

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Abstract

The (ring structure of the) Hochschild cohomology of the tensor product of two algebras was understood better thanks to Le and Zhou, who were able to express it in terms of the Hochschild cohomology of the two algebras. Using work by Grimley, Nguyen, and Witherspoon, as well as homotopy lifting techniques for Gerstenhaber brackets introduced by Volkov, we generalize Le and Zhou's result to some twisted tensor products. These have important applications in some quantum complete intersections also studied by Lopes and Solotar. This is joint work with Tolulope Oke and Sarah Witherspoon.

Contents

1 Hochschild cohomology, cup product, and Gerstenhaber bracket

Definition 1. *Let A be a k-algebra (our algebras are unital and associative, I'm not a monster. We define the Hochschild cohomology as):* $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$ *where* $A^e = A \otimes A^{op}$ *(is called the enveloping algebra of A). It comes with two operations (defined on cochains):*

$$\smile : \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \longrightarrow \mathrm{HH}^{m+n}(A),$$
$$[-,-] : \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \longrightarrow \mathrm{HH}^{m+n-1}(A).$$

These are called the cup product and the Gerstenhaber bracket. These operations, together with some compatibility conditions, make $\mathrm{HH}^*(A)$ into a Gerstenhaber algebra. This structure can be thought of as a graded Lie algebra. To define the cup and the bracket operations, the "bar resolution" is used, which is just a bunch of tensor products of A over the ground field. The cup product makes $\mathrm{HH}^*(A)$ into a graded commutative algebra, and now in the world of commutative things hopefully understanding this is easier. However, the payoff is that the Gerstenhaber bracket is complicated.

2 Le-Zhou's result, Grimley-Nguyen-Witherspoon's result, and techniques

Theorem 2 (Le-Zhou, 2014). *Let A and B be k-algebras, at least one of them finite dimensional. Then (as Gerstenhaber algebras):*

$$\mathrm{HH}^*(A \otimes B) \cong \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(B).$$

Remark 3. *It is a fact that if A and B are graded by the commutative groups F and G respectively, then $\mathrm{HH}^*(-)$ is bigraded: $\mathrm{HH}^{*,*}(-)$.*

Theorem 4 (Grimley-Nguyen-Witherspoon, 2017). *Let A and B be k-algebras, at least one of them finite dimensional, graded by the commutative groups F and G respectively. Then (as Gerstenhaber algebras):*

$$\mathrm{HH}^{*,F'\oplus G'}(A \otimes^t B) \cong \mathrm{HH}^{*,F'}(A) \otimes \mathrm{HH}^{*,G'}(B)$$

where $A \otimes^t B$ is the twisted tensor product by a bicharacter $t : F \otimes_{\mathbb{Z}} G \to k^\times$, and:

$$F' = \bigcap_{g\in G} \ker(t(-,g)), \quad G' = \bigcap_{f\in F} \ker(t(f,-)).$$

The k-vector space structure of $A \otimes^t B$ is given by $A \otimes B$, and the multiplication is given by taking $A \otimes B \otimes A \otimes B$, permuting the middle $B \otimes A$ and adding a scalar given by t, and then canonically multiplying together $A \otimes A$ and $B \otimes B$. You may complain, and rightfully so, that I have not told you how one can get a cup product or a Gerstenhaber bracket on a tensor product of Gerstenhaber algebras. For now, suffice to say that they exist and they satisfy what they should satisfy. The explicit expressions are:

$$(\alpha \otimes \beta) \smile (\alpha' \otimes \beta') = (-1)^{m'n}(\alpha \smile \alpha') \otimes (\beta \smile \beta'),$$
$$[\alpha \otimes \beta, \alpha' \otimes \beta'] = (-1)^{(m'-1)n}[\alpha, \alpha'] \otimes (\beta \smile \beta') + (-1)^{m'(n-1)}(\alpha \smile \alpha') \otimes [\beta, \beta'].$$

Remark 5. *The main tools used are (obviously more things are required, but let me focus on these two):*

1. *If $P \to A$ and $Q \to B$ are resolutions (of A and B bimodules respectively, satisfying some niceness conditions), then there is a chain (map) isomorphism (of A and B bimodules in each degree):*

$$\sigma : (P \otimes^t Q) \otimes_{A\otimes^t B} (P \otimes^t Q) \longrightarrow (P \otimes_A P) \otimes^t (Q \otimes_B Q).$$

2. *Alexander-Whitney and Eilenberg-Zilber maps.*

The first isomorphism is specially useful when the resolution and the twisting map are known, and a version of it that does not require inside knowledge of the twisting was generalized in [Karadag-McPhate-Oke-Ocal-Witherspoon]. The second maps are obnoxious to deal with, and they also require inside knowledge of the twisting, which means that using them to tackle the general case of a twisted tensor product is impossible.

3 Volkov's techniques and our application

Definition 6. *Given A a k-algebra, let $\mu_P : P \to A$ be a resolution of A-bimodules, $\Delta_P : P \to P \otimes_A P$ a diagonal map, and $\alpha \in \mathrm{Hom}_{A^e}(P_m, A)$ a cocycle. A homotopy lifting (of α with respect to Δ_P) is (an A-bimodule chain homomorphism) $\psi_\alpha : P \to P[1-m]$ such that:*

$$d(\psi_\alpha) = (\alpha \otimes 1_P - 1_P \otimes \alpha)\Delta_P, \quad \textit{and} \quad \mu_P \psi_\alpha \textit{ is cohomologous to } (-1)^{m-1}\alpha\psi$$

for some (A-bimodule chain map) $\psi : P \to P[1]$ for which $d(\psi) = (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$.

The point of presenting the complete definition is to show that the definition of homotopy lifting does not depend on a specific resolution, since such diagonal maps always exist. Moreover, Volkov proved that for any resolution, for any diagonal, and for any cocycle, homotopy liftings always exist! Moreover, they induce the Gerstenhaber bracket in cohomology! This is absolutely fantastic.

Theorem 7. *The bracket given at the chain level by:*

$$[\alpha, \beta] = \alpha\psi_\beta - (-1)^{(m-1)(n-1)}\beta\psi_\alpha$$

induces the Gerstenhaber bracket on Hochschild cohomology.

This method is inspired in results and work by Negron and Witherspoon, who in 2016 published what now is a special case of these homotopy liftings, where they focused on Koszul-like resolutions.

Lemma 8 (OOW). *In the twisted tensor product setup, let $P \to A$ and $Q \to B$ be resolutions of algebras (with the necessary finiteness conditions):*

$$\psi_{\alpha \otimes^t \beta} = \psi_\alpha \otimes^t (1_Q \otimes_B \beta)\Delta_Q + (-1)^m(\alpha \otimes_A 1_P)\Delta_P \otimes^t \psi_\beta$$

is a homotopy lifting of $\alpha \otimes^t \beta$ (in terms of homotopy liftings of α and β).

Theorem 9 (OOW). *As Gerstenhaber algebras (in the twisted tensor product setup, and assuming the necessary finiteness conditions, we have):*

$$\mathrm{HH}^{*,F' \oplus G'}(A \otimes^t B) \cong \mathrm{HH}^{*,F'}(A) \otimes \mathrm{HH}^{*,G'}(B).$$

Proof. Use the chain isomorphism σ as well as the Koszul sign convention (for both the Lemma and the Theorem). □

By expanding the conditions of Volkov's homotopy lifting, being careful with what constitutes a diagonal for the twisted tensor product of resolutions, and checking that the Gerstenhaber bracket that I have not explicitly given you on the right hand side coincides with what is coming from the left. This allows computing the Gerstenhaber bracket in the Hochschild cohomology of a twisted tensor product $A \otimes^t B$, a notoriously difficult task, as long as we know the Gerstenhaber bracket in the respective Hochschild cohomologies of A and B. This has applications in, for example, deformations of algebras.

4 Remarks and future work

1. We did not use the explicit formula for σ (at least not its full expression).

   The original proofs required the explicit expression of σ because of the use of the Alexander-Whitney and Eilenberg-Zilber maps, but we only used that σ makes some diagrams commute. This should also hold for the version in [KMOOW], and is current work in progress.

2. Compute more examples.

   New examples and complete computations are always useful, the current examples are relatively small and relatively scarce.

3. Understand why some examples (like the Jordan plane) work: $k\langle x, y\rangle/(yx - xy - x^2)$.

   The complete Gerstenhaber algebra structure of the Jordan plane was first computed by Lopes and Solotar, using spectral sequences and a lot of machinery. In [KMOOW] we also computed it using more elementary and completely different methods; and although the hypothesis that we required on the twisting map were not satisfied, using these elementary techniques the conclusions of our main results held. That is, applying our constructions, we were still able to compute the complete Gerstenhaber algebra structure. What are then the correct hypothesis on the twist?

Thank you for your time!

References

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