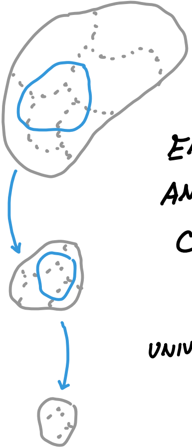
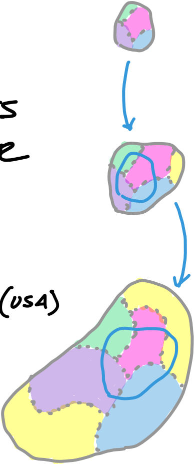


ENRICHED COVERAGES AND SHEAVES UNDER CHANGE OF BASE

ARI ROSENFIELD
UNIV. MARYLAND – COLLEGE PARK (USA)



OVERVIEW

- PREPRINT ON arXiv AT 2405.19529

- MAIN CONTRIBUTION: "WHEN DOES CHANGE OF ENRICHING BASE CATEGORY INTERACT NICELY WITH ENRICHED COVERAGES? WHAT ABOUT ENRICHED SHEAFIFICATION?"

  A. WHEN BASE CHANGE IS EFFECTED BY A FAITHFUL, CONSERVATIVE RIGHT MONOIDAL ADJOINT.

OVERVIEW

OKAY, WHAT'S BASE CHANGE, WHAT'S A COVERAGE, AND WHAT DO THEY HAVE TO DO WITH SPECTRA?

OVERVIEW

IF YOU'RE IN THIS ROOM, YOU'VE PROBABLY WORKED WITH ENRICHED CATEGORIES: CAT'S WHOSE HOMS HAVE MORE GENERAL STRUCTURE THAN THAT OF A SET.

ENRICHED FUNCTORS / NATURAL TRANSFORMATIONS RESPECT THIS EXTRA STRUCTURE.

OVERVIEW

e.g., ANY PREADDITIVE CATEGORY IS ENRICHED OVER Ab (A FORTIORI, SO IS ANY ADDITIVE OR ANY ABELIAN CAT.)

e.g., A RING IS AN Ab-CATEGORY WITH ONE OBJECT; A RING HOMOMORPHISM IS AN Ab-FUNCTOR

e.g., A K-ALGEBRA (comm. RING k) IS A Mod_k-CATEGORY WITH ONE OBJECT,

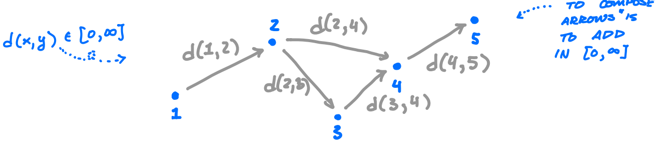
ETC...

OVERVIEW

DIFFERENT CLASS OF EXAMPLE:

e.g., ENRICH OVER THE MONOIDAL PREORDER $([0,\infty], \geq, +, 0)$ TO GET A LAWVERE METRIC SPACE.

(ENCOMPASSES PLAIN OLD METRIC SPACES)



A $[0,\infty]$-FUNCTOR BETWEEN SUCH IS A 1-LIPSCHITZ FUNCTION.

OVERVIEW

WEAKER / "UNSATURATED" TOPOLOGY

GROTHENDIECK TOPOLOGIES AND COVERAGES ARE WAYS TO ENDOW GENERAL CATEGORIES (NOT JUST $\mathcal{O}(X)$ FOR $X \in \mathrm{Top}$) WITH A NOTION OF "WHICH OBJECTS COVER WHICH."

THIS IS SO WE CAN DEFINE SHEAVES ON CATEGORIES.

OVERVIEW

THERE ARE ENRICHED ANALOGUES OF TOPOLOGIES, COVERAGES, AND SHEAVES FOR ENRICHED CATEGORIES.

CLASSICAL CASES (SEE TEXTBOOK ACCOUNTS BY STENSTRÖM, NASTASESCU) WHEN ENRICHMENT IS TAKEN OVER Ab OR $grMod_k$.

OVERVIEW

e.g., AN Ab-TOPOLOGY ON A RING R IS THE SAME THING AS A GABRIEL LOCALIZING SYSTEM OF RIGHT IDEALS ON R; AN Ab-SHEAF ON R W/R/T SUCH IS THE GABRIEL LOCALIZATION OF R.

(MORE ON THIS LATER)

NOTATION

$(\mathcal{U}, \otimes, *_{\mathcal{U}})$ AND $(\mathcal{V}, \otimes, *_{\mathcal{V}})$ ARE ALWAYS SYMMETRIC MONOIDAL CLOSED AND LOCALLY SMALL.

$\mathcal{C}$ IS ALWAYS AN ESSENTIALLY SMALL $\mathcal{V}$-ENRICHED CAT. ($\mathcal{V}$-CAT.)

OVERVIEW

FROM A CATEGORY-THEORETIC PERSPECTIVE, CATEGORIES OF SHEAVES (i.e., TOPOSES!) ARE VERY NICE,

e.g., CARTESIAN CLOSED,
BICOMPLETE, ETC...

SO THE FOLLOWING THEOREM IS SUFFICIENT MOTIVATION FOR ME TO CARE ABOUT COVERAGES:

OVERVIEW

THEOREM 0. (QUINTEIRO – BORCEUX, 1996)

THERE ARE BIJECTIVE CORRESPONDENCES

$$\{\text{ENRICHED TOPOLOGIES ON } \mathcal{C}\}$$

$$\longleftrightarrow$$

(*) $$\{\text{UNIVERSAL CLOSURE OPERATIONS ON } [\mathcal{C}^{op}, \mathcal{V}]\}$$

$$\longleftrightarrow$$

$$\{\text{REFLECTIVE } \mathcal{V}\text{-SUBCATS OF } [\mathcal{C}^{op}, \mathcal{V}]\}$$

↳ AKA $\mathcal{V}$-SHEAF TOPOSES.

" STUDYING CATEGORIES OF ENRICHED SHEAVES IS STUDYING GROTHENDIECK TOPOLOGIES. "

OVERVIEW

IN THE CASE $\mathcal{V} = Ab$, $\mathcal{A}$ A RING, THE ANALOGUE OF (0) READS:

THEOREM (GABRIEL, MARANDA - 1960s)

THERE ARE BIJECTIVE CORRESPONDENCES

($\mathcal{T}, \mathcal{F}$) WITH $\mathcal{T}$ CLOSED UNDER SUBMOD'S

$$\{\text{GABRIEL TOPOLOGIES ON } \mathcal{A}\}$$
$$\longleftrightarrow$$
$$\{\text{HEREDITARY TORSION THEORIES FOR } \mathcal{A}\}$$
$$\longleftrightarrow$$
$$\{\text{LEFT EXACT RADICALS OF } \mathrm{Mod}_{\mathcal{A}}\}$$

"NICE" SUBFUNCTORS OF $id_{\mathrm{Mod}\,\mathcal{A}}$

OVERVIEW

SO STUDYING Ab - TOPOLOGIES ON $\mathcal{A}$ "IS" STUDYING HEREDITARY TORSION THEORIES ON $\mathcal{A}$,

AND STUDYING THE ENRICHED CATEGORY $[\mathcal{C}^{op}, \mathcal{V}]$ "IS" STUDYING "Mod-$\mathcal{C}$."

OVERVIEW

SO THAT'S WHAT A COVERAGE IS.

WHAT'S BASE CHANGE?

THINKING OF $\mathcal{U}$ & $\mathcal{V}$ AS TWO DIFFERENT TYPES OF STRUCTURE THE HOMS IN $\mathcal{C}$ MIGHT HAVE, WE CAN CHANGE THE BASE OF $\mathcal{C}$ VIA A MONOIDAL FUNCTOR $G: \mathcal{V} \longrightarrow \mathcal{U}$, OBTAINING A $\mathcal{U}$-CATEGORY $G_* \mathcal{C}$.

OVERVIEW

e.g., IF $\mathcal{V} = Ab$ (IN FACT, IF $\mathcal{V}$ IS ANY CONCRETE CATEGORY), $\mathcal{C}$ HAS AN UNDERLYING $\overset{\text{Set-}}{\vee}$CATEGORY $\mathcal{C}_0$, OBTAINED BY BASE CHANGE VIA

$$G = Hom_{Ab}(\mathbb{Z}, -) : Ab \longrightarrow Set .$$

OVERVIEW

SO, THE RESULTS TO FOLLOW ADDRESS HOW "ENRICHED SHEAF THEORY" INTERACTS WITH BASE CHANGE.

OKAY BUT STILL, WHAT'S THAT GOT TO DO WITH SPECTRA ??

OVERVIEW

WE JUST SAW A BUNCH OF RESULTS OBSTRUCTING EXTENSION OF FUNCTORIAL SPECTRA TO CAT'S OF NONCOMMUTATIVE ALGEBRAS.

IN MANNY'S TALK!

OVERVIEW

THE DREAM SCENARIO TO CIRCUMVENT ONE OF THESE IS FINDING A SYMMETRIC MONOIDAL CAT. "qSet" AND A COMMUTING DIAGRAM

$$\begin{tikzcd}
\mathrm{commRing}^{op} \arrow[r, "\mathrm{Spec}"] \arrow[d] & \mathrm{Set} \arrow[d, dashed] \\
\mathrm{Ring}^{op} \arrow[r, dotted, "\text{"ncSpec"}"'] & \mathrm{qSet}
\end{tikzcd}$$

OVERVIEW

HOW TO TELL IF A GIVEN CANDIDATE ν FOR q_{Set} IS THE RIGHT ONE?

OVERVIEW

FOR SUCH A $\mathcal{V}$, THERE SHOULD BE $\mathcal{V}$-ENRICHED COVERAGES ON A GIVEN $\mathcal{C}$ WHICH DON'T ARISE FROM $\mathbf{Set}$-ENRICHED ONES: i.e., IF WE CHANGE BASE FROM $\mathbf{Set} \longrightarrow \mathcal{V}$, WE SHOULD HAVE AN INJECTION

$$\{\mathbf{Set}\text{-COVERAGES}\} \hookrightarrow \{\mathcal{V}\text{-COVERAGES}\}.$$

A PROBLEM : A PLEA

SO THAT'S WHY I'M DOING ALL THIS.

BUT WE HAVEN'T FOUND $q\mathsf{Set}$, YET.

I HAVEN'T BEEN ABLE TO COME UP WITH MANY CONCRETE APPLICATIONS OF THESE (AS YOU'LL SEE, TECHNICAL CATEGORY-THEORETIC) RESULTS THAT AREN'T THE HYPOTHETICAL ONE I DESCRIBED.

A PROBLEM : A PLEA

JOURNAL EDITORS DON'T LIKE THAT!

SO I'M VERY INTERESTED IN HEARING ABOUT LITERALLY ANY IDEAS YOU ALL MIGHT HAVE FOR WHAT TO DO WITH THIS STUFF

NOW, THE DETAILS..

$\mathcal{V}$ - ENRICHED CATEGORIES

REMINDER. A $\mathcal{V}$-CATEGORY $\mathcal{C}$ IS A COLLECTION OF OBJECTS WITH, $\forall x, y, z \in Ob(\mathcal{C})$,

- AN OBJECT $\mathcal{C}(x,y) \in \mathcal{V}$
- A DISTINGUISHED MORPHISM
$$id_x : *_{\mathcal{V}} \longrightarrow \mathcal{C}(x,x) \in \mathcal{V}$$
- A MORPHISM
$$\circ_{xyz} : \mathcal{C}(y,z) \otimes \mathcal{C}(x,y) \longrightarrow \mathcal{C}(x,z)$$

WHICH ARE ASSOCIATIVE AND UNITAL.

(i.e., OBVIOUS DIAGRAMS IN $\mathcal{V}$ COMMUTE, MAYBE INVOLVING MONOIDAL COHERENCE MAPS IN $\mathcal{V}$.)

$\mathcal{V}$ - ENRICHED CATEGORIES

REMINDER. A $\mathcal{V}$-FUNCTOR $R : \mathcal{C} \to \mathcal{D}$ BETWEEN $\mathcal{V}$-CATEGORIES IS AN ASSIGNMENT $x \longmapsto Rx$ ON OBJECTS AND, FOR EACH PAIR $x, y \in Ob(\mathcal{C})$, A MORPHISM

$$R_{xy} : \mathcal{C}(x,y) \longrightarrow \mathcal{D}(Rx, Ry) \in \mathcal{V}$$

SATISFYING "OBVIOUS" COMMUTING DIAGRAMS.

$\mathcal{V}$ - ENRICHED CATEGORIES

REMINDER. $\mathcal{V}$-Cat IS A 2-CATEGORY.

$\mathcal{V}$ - ENRICHED CATEGORIES

ALREADY HEARD ABOUT EXAMPLES WITH $\mathcal{V} = Ab,\ grMod_R,\ [0,\infty]$. TWO MORE:

- $\mathcal{V} = \{0,1\}$: A $\mathcal{V}$-CATEGORY IS A PREORDERED SET;
- $\mathcal{V} = Cat$: A $\mathcal{V}$-CATEGORY IS A STRICT 2-CATEGORY.

$\mathcal{V}$-ENRICHED CATEGORIES

WE'LL ONLY CONSIDER ENRICHMENT OVER $\mathcal{U}, \mathcal{V}$ WHICH ARE

1. LOCALLY FINITELY PRESENTABLE:

EVERY OBJECT IS A FILTERED COLIMIT OF COMPACT OBJECTS. ($x \in \mathrm{Ob}\,\mathcal{V}$ IS **COMPACT** IF $\mathrm{Hom}_{\mathcal{V}}(x,-)$ PRESERVES FILTERED COLIMITS.)

e.g., $\mathrm{Cpt}(\mathrm{Set}) = \mathrm{FinSet}$,

$\mathrm{Cpt}(\mathrm{Ab}) =$ f.p. ABELIAN GRPS, ...

$\mathcal{V}$ - ENRICHED CATEGORIES

WE'LL ONLY CONSIDER ENRICHMENT OVER $\mathcal{U}, \mathcal{V}$ WHICH ARE

1. LOCALLY FINITELY PRESENTABLE:

WANT COMPACTS TO GENERATE $\mathcal{V}$ UNDER COLIMITS SO WE CAN SPEAK SENSIBLY OF "ELEMENTS" (GENERALIZED ELEMENTS) OF $\mathcal{C}(x,y) \in \mathcal{V}$. (NOTE $*_{\mathcal{V}}$ MAY NOT BE A "SINGLE POINT" IN GENERAL.)

$\mathcal{V}$ - ENRICHED CATEGORIES

WE'LL ONLY CONSIDER ENRICHMENT OVER $\mathcal{U}, \mathcal{V}$ WHICH ARE

1. LOCALLY FINITELY PRESENTABLE:

   DENOTE THE SET OF COMPACT GENERATORS OF $\mathcal{V}$ BY $\mathcal{G}_{\mathcal{V}}$.

$\mathcal{V}$-ENRICHED CATEGORIES

WE'LL ONLY CONSIDER ENRICHMENT OVER $\mathcal{U}, \mathcal{V}$ WHICH ARE

1. LOCALLY FINITELY PRESENTABLE:

   FACT. ANY LFP CATEGORY IS COCOMPLETE AND WELL-POWERED

   ($\mathrm{Sub}(x)$ IS ALWAYS A SMALL SET.)

$\mathcal{V}$-ENRICHED CATEGORIES

WE'LL ONLY CONSIDER ENRICHMENT OVER $\mathcal{U}, \mathcal{V}$ WHICH ARE

2. FINITELY COMPLETE.

LATER, WILL ADD:

3. BARR-REGULAR (AND EXPLAIN WHY!)

$\mathcal{V}$ - ENRICHED CATEGORIES

I'LL CALL A SymMonCat SATISFYING ① AND ② A **COSMOS**.

e.g., Set, Ab, Mod_R, grMod_R, Cat, $\{0,1\}$

BUT SADLY, NOT $[0,\infty]$: FOR POSETS, LFP $\iff$ ALGEBRAIC LATTICE, AND IT'S NOT ALGEBRAIC. MANY CONSTRUCTIONS BELOW STILL WORK FOR $[0,\infty]$, SO I KEEP IT AROUND!

BASE CHANGE

WE NEED SOME TECHNICAL DETAILS ON BASE CHANGE, TOO.

<u>FACT:</u> ANY MONOIDAL FUNCTOR

$$G : \mathcal{V} \longrightarrow \mathcal{U}$$

INDUCES A 2-FUNCTOR

$$G_* : \mathcal{V}\text{-}\mathrm{Cat} \longrightarrow \mathcal{U}\text{-}\mathrm{Cat},$$

CALLED **CHANGE OF BASE**.

BASE CHANGE

NOTATION:

$G_*\mathcal{C}$ IS A $\mathcal{U}$-CATEGORY WITH HOMS

$$G_*\mathcal{C}(x,y) := G(\mathcal{C}(x,y)).$$

$$(\mathrm{Ob}(G_*\mathcal{C}) := \mathrm{Ob}(\mathcal{C}).)$$

IF $R : \mathcal{C} \to \mathcal{D}$ IS A $\mathcal{V}$-FUNCTOR WITH EFFECT $R_{xy} : \mathcal{C}(x,y) \to \mathcal{D}(Rx, Ry) \in \mathcal{V}$,

G_*R IS A $\mathcal{U}$-FUNCTOR

$$G_*\mathcal{C} \longrightarrow G_*\mathcal{D}$$

WITH EFFECT $GR_{xy} \in \mathcal{U}$.

BASE CHANGE

BEING A 2-FUNCTOR, G_* ALSO AFFECTS $\mathcal{V}$-NATURAL TRANSFORMATIONS, BUT IT WOULD PUT YOU TO SLEEP.

SEE PREPRINT FOR THE GUTS & GORE.

BASE CHANGE

WE'LL ALWAYS TAKE $G : \mathcal{V} \to \mathcal{U}$ TO BE

i. LAX MONOIDAL

ii. FAITHFUL

iii. CONSERVATIVE

($Gf : Gx \to Gy$ ISO $\Rightarrow$ $f : x \to y$ ISO)

iv. A MONOIDAL RIGHT ADJOINT

$$\begin{tikzcd} \mathcal{U} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{V} \arrow[l, bend left, "G"] \end{tikzcd} .$$

BASE CHANGE

FACT. (KELLY, 1974)

MONOIDAL LEFT ADJOINTS ARE STRONG MONOIDAL.

(OUR NOTATION: $F(x \otimes y) \cong F(x) \otimes F(y)$,
VS. $F(x) \otimes F(y) \longrightarrow F(x \otimes y)$.)

BASE CHANGE

EXAMPLES!

FORGETFUL FUNCTORS ON $(G \circ F)$-Mod FOR MONADS $G \circ F$

(1) MONADIC FUNCTORS:

- FOR $(f : R \to S) \in \mathrm{commRing}$, RESTRICTION OF SCALARS
$$f^* : S\mathrm{Mod} \longrightarrow R\mathrm{Mod}$$
- FORGETFUL FUNCTORS $\mathrm{Mon}(\mathcal{V}) \to \mathcal{V}$
- REFLECTIVE SUBCATEGORY INCLUSIONS
- NERVE $\mathrm{Cat} \xrightarrow{N} s\mathrm{Set}$

BASE CHANGE

"ALMOST EXAMPLES"

(INVOLVING NON-LFP CAT'S)

(1) $\{0,1\} \hookrightarrow [0,1]$

(2) $-\log(*) : [0,1] \rightleftarrows [0,\infty] : e^{-(*)}$

SOME RESULTS WILL STILL HOLD FOR THESE, OTHERS WON'T.

COVERAGES: FIRST PASS

GOAL. APPLY BASE CHANGE TO A $\mathcal{V}$-COVERAGE ON $\mathcal{C}$.

WHAT IS A $\mathcal{V}$-COVERAGE?

COVERAGES: FIRST PASS

(QUINTEIRO - BORCEUX, 1996)

DEF. A $\mathcal{V}$-COVERAGE ON $\mathcal{C}$ IS, ?? ??

$\forall\, x \in Ob(\mathcal{C})$, A FAMILY $J(x) \subseteq Sub(\mathcal{C}(-,x))$

SUCH THAT:

(T1) $\mathcal{C}(-,x) \in J(x)$

(T2) $\forall R \in J(x)$ AND $f \in Hom_{\mathcal{V}}(G_{\mathcal{V}}, \mathcal{C}(y,x))$,

THE PULLBACK $R_f \in J(y)$.

IN $[\mathcal{C}^{op}, \mathcal{V}]$

COVERAGES: FIRST PASS

DEF. IF A $\mathcal{D}$-COVERAGE J ON $\mathcal{C}$ SATISFIES $(\forall x \in \mathcal{C})$

(T3) IF $R \in \mathrm{Sub}(\mathcal{C}(-,x))$ AND $S \in J(x)$ IS SUCH THAT $\forall y,\ \forall f \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{Y}_{\mathcal{D}}, S(y))$

$R_f \in J(y)$, THEN $R \in J(x)$

THEN J IS CALLED A **TOPOLOGY**.

THESE CAUSED PROBLEMS: MORE ON THAT LATER!

COVERAGES: FIRST PASS

e.g., IF $\mathcal{C}$ IS A RING (Ab-CAT WITH ONE OBJECT), AN Ab-PRESHEAF $\mathcal{P}: \mathcal{C}^{op} \longrightarrow Ab$ IS A RIGHT $\mathcal{C}$-MODULE.

$\mathcal{C}(-,\bullet)$ IS THE $\mathcal{C}$-MODULE $\mathcal{C}_{\mathcal{C}}$, AND A SUBOBJECT $R \rightarrowtail \mathcal{C}(-,\bullet)$ IS A RIGHT IDEAL OF $\mathcal{C}$.

COVERAGES: FIRST PASS

e.g. A GABRIEL TOPOLOGY ON A RING $\mathcal{C}$ IS A FAMILY $\mathcal{J}$ OF RIGHT IDEALS OF $\mathcal{C}$ SUCH THAT

(T1) $\mathcal{C} \in \mathcal{J}$

(T2) $\forall f \in \mathcal{C},\ I \in \mathcal{J},\quad (I : f) \in \mathcal{J}$

+ ANALOGUE OF (T3)

BASE CHANGE FOR PRESHEAVES

SO, TO APPLY BASE CHANGE TO A COVERAGE, WE'LL NEED TO APPLY IT TO PRESHEAVES FIRST.

BASE CHANGE FOR PRESHEAVES

FOR A $\mathcal{V}$- PRESHEAF $R : \mathcal{C}^{op} \to \mathcal{V}$,

$$G_* R : G_* \mathcal{C}^{op} \longrightarrow G_* \mathcal{V}$$

IS NOT A $\mathcal{U}$- PRESHEAF : IT DOESN'T TAKE VALUES IN $\mathcal{U}$.

WE CAN FIX THIS :

BASE CHANGE FOR PRESHEAVES

FACT. $F \dashv G \in \mathrm{Cat}$ CANONICALLY INDUCES AN ADJUNCTION

$$F^{\mathcal{U}} \dashv G^{\mathcal{U}} : G_* \mathcal{V} \rightleftarrows \mathcal{U}$$

IN $\mathcal{U}$-Cat. (ULTIMATELY SINCE F IS STRONG MONOIDAL.)

DENOTE

$$\phi_{ab} : G\mathcal{V}(F^{\mathcal{U}}a, b) \xrightarrow{\sim} \mathcal{U}(a, G^{\mathcal{U}}b).$$

BASE CHANGE FOR PRESHEAVES

CONSTRUCTION (R.)

DEFINE A $\mathcal{U}$-FUNCTOR $\tilde{G}R$ BY

$x \longmapsto GRx$ ON OBJECTS AND

$$\tilde{G}R_{xy} := \phi \circ (G_* R)_{xy}$$

ON HOMS.

BASE CHANGE FOR PRESHEAVES

FINALLY WE CAN STATE SOME RESULTS!

THEOREM 1 (R.

THE $\tilde{G}$ CONSTRUCTION IS FUNCTORIAL

$$\tilde{G}(-) : [\mathcal{C}^{op}, \mathcal{V}] \longrightarrow [G_* \mathcal{C}^{op}, \mathcal{U}],$$

PRESERVES LIMITS,

AND MOREOVER IS FAITHFUL AND

CONSERVATIVE IF G IS.

BASE CHANGE FOR PRESHEAVES

SO, WE NOW HAVE A WELL-DEFINED MAP

$$\mathrm{Mono}_{\mathcal{V}}(\mathcal{C}(-,x)) \xrightarrow{\tilde{G}} \mathrm{Mono}_{\mathcal{U}}(\hat{G}\mathcal{C}(-,x)).$$

$\mathcal{V}$-MONICS WITH TARGET $\mathcal{C}(-,x)$.

(JUST B/C $\tilde{G}$ IS LIMIT-PRESERVING.)

IS IT WELL-DEFINED ON ISOMORPHISM CLASSES IN $\mathrm{Mono}_{\mathcal{V}}(\mathcal{C}(-,x))$, THOUGH?

THEOREM (R.) YES, IT IS! (SEE PAPER ü)

BASE CHANGE FOR PRESHEAVES

SO, WE HAVE THE FOLLOWING FOR PRESHEAVES:

THEOREM 2 (R.) FOR G FAITHFUL AND CONSERVATIVE,

$$\mathrm{Sub}_{\mathcal{V}}(\mathcal{C}(-,x)) \xrightarrow{\tilde{G}} \mathrm{Sub}_{\mathcal{U}}(\tilde{G}\mathcal{C}(-,x))$$

IS INJECTIVE AND MONOTONE.

A "CUTE" APPLICATION

BEFORE MOVING ON TO THE MAIN THEOREM, NOTE THAT 2 HAS AN EASY COROLLARY.

PERSPECTIVE: $\mathcal{P}: \mathcal{C}^{op} \to \mathcal{V}$ IS A $\mathcal{V}$-REPRESENTATION OF $\mathcal{C}$ ON $\mathcal{V}$.

IT HAS FINITE LENGTH IF IT SATISFIES ACC + DCC ON SUBOBJECTS.

COROLLARY. IF $\tilde{G}\mathcal{P} \in [G_*\mathcal{C}^{op}, \mathcal{U}]$ HAS FINITE LENGTH, THEN $\mathcal{P} \in [\mathcal{C}^{op}, \mathcal{V}]$ HAS FINITE LENGTH.

NOT SURPRISING, BUT PERHAPS USEFUL IN ITS GENERALITY!

BASE CHANGE FOR COVERAGES

TURNING TO COVERAGES & TOPOLOGIES, WE'D HOPE THAT THE COLLECTIONS OF ALL SUCH ARE COMPLETE LATTICES.

THEOREM (R.) THEY ARE!

DENOTE

$$\Sigma(\mathcal{C}, \mathcal{V}) := \{\mathcal{V}\text{-COVERAGES ON } \mathcal{C}\}$$

$$\tau(\mathcal{C}, \mathcal{V}) := \{\mathcal{V}\text{-TOPOLOGIES ON } \mathcal{C}\}$$

BASE CHANGE FOR COVERAGES

THERE'S NOW NO SUBTLETY IN APPLYING THE $\tilde{G}$ CONSTRUCTION TO A $\mathcal{V}$-COVERAGE — THE NAÏVE THING WORKS:

PROPOSITION. FOR $J \in \Sigma(\mathcal{C}, \mathcal{V})$, WE HAVE $\tilde{G}J \in \Sigma(G_*\mathcal{C}, \mathcal{U})$.

BASE CHANGE FOR COVERAGES

MOREOVER ...

THEOREM 4 (R.)

$$\Sigma(\mathcal{C}, \mathcal{V}) \xrightarrow{\tilde{G}} \Sigma(G_*\mathcal{C}, \mathcal{U})$$

IS AN INJECTIVE LATTICE MORPHISM.

THAT'S WHAT WE WANTED, OR CLOSE TO IT!

BASE CHANGE FOR TOPOLOGIES?

WHAT ABOUT TOPOLOGIES? I'M NOT SURE! PROBABLY IS A VERSION OF 4 AND IN FACT $\tilde{G}$ DOES INDUCE A TOPOLOGY $\overline{GJ}$ GIVEN A COVERAGE J (TRANSFINITE CONSTRUCTION), BUT I WOULD NEED A VERY CREATIVE NEW METHOD TO PROVE:

CONJECTURE.

$$\mathcal{T}(\mathcal{C}, \mathcal{V}) \xrightarrow{\overline{G}} \mathcal{T}(G_*\mathcal{C}, \mathcal{U})$$

IS AN INJECTIVE LATTICE MORPHISM.

BASE CHANGE FOR TOPOLOGIES?

(UNFORTUNATELY THIS LAST PROBLEM WAS SO HARD THAT I WASTED SEVERAL MONTHS OF MY Ph.D. TRYING TO SOLVE IT INSTEAD OF FINDING APPLICATIONS OF THE COVERAGES RESULT! NEVERTHELESS, HERE WE ARE.)

GRADED ALGEBRAS: A COUNTEREXAMPLE

DOES (4) HOLD IF G IS ONLY A RIGHT ADJOINT? IT TURNS OUT THAT THE ANSWER IS NO!

EXAMPLE. FOR k A FIELD, SET $\mathcal{V} = \mathrm{grMod}_k$; AND SET

RIGHT ADJOINT, BUT NEITHER FAITHFUL NOR CONS.

$$G = \mathrm{Hom}_{\mathcal{V}}(k, -) : \mathcal{V} \longrightarrow \mathrm{Set}.$$

(G PICKS OUT THE DEGREE-0 ELEMENTS OF A GIVEN $X \in \mathrm{Ob}(\mathcal{V})$.)

GRADED ALGEBRAS

FACT. ANY MULTIPLICATIVELY CLOSED SET $S \subseteq X$ OF HOMOGENEOUS ELEMENTS GIVES RISE TO A $\mathcal{D}$-TOPOLOGY ON $X \in grAlg_K$ VIA

$$H_S = \{ I \triangleleft X : (I:x) \cap S \neq \emptyset \quad \forall x \in X \text{ HOMOGENEOUS} \}$$

GRADED ALGEBRAS

TAKE $X = k[x,y]$, $S = \{x^n\}$, $T = \{y^n\}$.

THE γ-COVERAGES

$$H_S = \{ I \triangleleft A : \ I \text{ HOMOG. } \dot{\div} \ \exists n \text{ s.t. } x^n \in I \}$$

$$H_T = \{ I \triangleleft A : \ I \text{ HOMOG. } \dot{\div} \ \exists n \text{ s.t. } y^n \in I \}$$

ARE DISTINCT, BUT SINCE $\tilde{G} I = Hom_k(k, I_0)$,

AND $I_0 = \{0\}$ OR k ALWAYS,

$$\tilde{G} H_S = \{ (0), k \} = \tilde{G} H_T .$$

GRADED ALGEBRAS

SO: IF 4 HOLDS, G MUST BE FAITHFUL OR CONSERVATIVE.

RETURNING TO THIS PROJECT AFTER MANY MONTHS, THIS MAKES ME WONDER:

GRADED ALGEBRAS

OPEN QUESTION. IS THERE A CHARACTERIZATION OF MONADIC LAX MONOIDAL FUNCTORS LURKING IN THIS WORK SOMEWHERE?

(IF (4) HOLDS, MUST G BE MONADIC? HMM...)

BASE CHANGE FOR SHEAVES

ANYWAY! LAST THING TO NOTE:

ASSUMING $\mathcal{V}$ IS "BARR-REGULAR", SO $[\mathcal{C}^{op}, \mathcal{V}]$ IS REGULAR AS AN UNENRICHED CAT.,

THERE IS A NOTION OF ENRICHED "SHEAFIFICATION" DUE TO QUINTEIRO-BORCEUX. I'LL SPARE THE DETAILS OF THE CONSTRUCTION: IT'S A $\mathcal{V}$-ENDOFUNCTOR

$$\Sigma\Sigma : [\mathcal{C}^{op}, \mathcal{V}] \longrightarrow [\mathcal{C}^{op}, \mathcal{V}] .$$

BASE CHANGE FOR SHEAVES

IF WE ASSUME G IS FULL, WE HAVE THE FOLLOWING:

THEOREM 5.

FOR $P \in [\mathcal{C}^{op}, \mathcal{V}]$, $\tilde{G}(\Sigma\Sigma P) \cong \Sigma\Sigma \tilde{G} P$.

ALSO NOT SURPRISING!

THAT'S ALL!

THANKS FOR LISTENING!