Representation theory of generalized Weyl–Heisenberg groups: Structure, classification, and application

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Preliminaries and Notation

Let H and K be two locally compact groups with the identity elements e_H and e_K, respectively and let $\tau : H \to Aut(K)$ be a homomorphism such that the map $(h,k) \mapsto \tau_h(k)$ is continuous from $H \times K$ onto K, where $H \times K$ equips with the product topology. The semi- direct product topological group $G_\tau = H \times_\tau K$ is the locally compact topological space $H \times K$ under the product topology, with the group operations:

$$(h_1, k_1) \times_\tau (h_2, k_2) = (h_1 h_2, k_1 \tau_{h_1}(k_2),$$

$$(h,k)^{-1} = (h^{-1}, \tau_{h^{-1}}(k^{-1})).$$

Introduction

It is worth to note that $K_1 = \{(e_H, k); k \in K\}$ is a closed normal subgroup and $H_1 = \{(h, e_K); h \in H\}$ is a closed subgroup of G_τ such that $G_\tau = HK$. Moreover, the left Haar measure of the locally compact group G_τ is

$$d\mu_{G_\tau}(h,k) = \delta_H(h) d\mu_H(h) d\mu_K(k),$$

in which $d\mu_H, d\mu_K$ are the left Haar measures on H and K, respectively and $\delta_H : H \to (0, \infty)$ is a positive continuous homomorphism that satisfies

$$d\mu_K(k) = \delta_H(h) d\mu(\tau_h(k)),$$

for $h \in H, k \in K$.

Introduction

Moreover, the modular function Δ_{G_τ} is

$$\Delta_{G_\tau} = \delta_H(h)\Delta_H(h)\Delta_K(k),$$

where Δ_H, Δ_K are the modular functions of H, K, respectively. When K is also abelian, one can define $\hat{\tau} : H \to Aut(\hat{K})$ via $h \mapsto \hat{\tau_h}$ where

$$\hat{\tau_h}(\omega) = \omega \circ \tau_{h^{-1}},$$

for all $\omega \in \hat{K}$. We usually denote $\omega \circ \tau_{h^{-1}}$ by ω_h. With this notation, it is easy to see

$$\omega_{h_1 h_2} = (\omega_{h_2})_{h_1},$$

where $h_1, h_2 \in H$ and $\omega \in \hat{K}$.

Introduction

The semi-direct product $G_{\hat{\tau}} = H \times_{\hat{\tau}} \hat{K}$ is a locally compact group with the left Haar measure

$$d\mu_{\hat{G}}(h,\omega) = \delta_H(h)^{-1} d\mu_H(h) d\mu_{\hat{K}}(\omega),$$

where $d\mu_{\hat{K}}$ is the Haar measure on $\hat{K}$. Also, for all $h \in H$,

$$d\mu_{\hat{K}}(\omega_h) = \delta_H(h) d\mu_{\hat{K}}(\omega),$$

for $\omega \in \hat{K}$, (see more details in [4, 1, 3].)

Introduction

Let $G_\tau = H \times_\tau K$, and define $\theta : G_\tau \to Aut(\hat{K} \times \mathbb{T})$ via

$$(h, k) \mapsto \theta_{(h,k)}(\omega, z) = (\hat{\tau_h}(\omega), \hat{\tau_h}(\omega)(k)z) = (\omega_h, \omega_h(k)z),$$

for all $(h, k) \in H \times_\tau K$ and $(\omega, z) \in \hat{K} \times \mathbb{T}$. The mapping θ is a continuous homomorphism. Thus the semi-direct prodoct

$$G_\tau \times_\theta (\hat{K} \times \mathbb{T}) = (H \times_\tau K) \times_\theta (\hat{K} \times \mathbb{T}),$$

is a locally compact group and it is called the generalized Weyl Heisenberg group associated with the semi direct product group $G_\tau = H \times_\tau K$, and denoted by $\mathbb{H}(G_\tau)$.

Introduction

It is easy to see that the group operations of $\mathbb{H}(G_\tau)$ are

$$(h_1, k_1, \omega_1, z_1).(h_2, k_2, \omega_2, z_2) = (h_1 h_2, k_1 \tau_{h_1}(k_2), \omega_1 \omega_{2h_1}, \omega_{2h_1}(k) z_1 z_2),$$

$$(h_1, k_1, \omega_1, z_1)^{-1} = (h_1^{-1}, \tau_{h_1}^{-1}(k^{-1}), \bar{\omega}_{h_1^{-1}}, \bar{\omega}_{h_1^{-1}}(\tau_{h_1}^{-1}(k^{-1}))z^{-1}),$$

for $(h_1, k_1, \omega_1, z_1), (h_2, k_2, \omega_2, z_2) \in \mathbb{H}(G_\tau)$ (see [4]) and the left Haar measure of $\mathbb{H}(G_\tau)$ is:

$$d\mu_{\mathbb{H}(G_\tau)}(h, k, \omega, z) = d\mu_H(h) d\mu_K(k) d\mu_{\hat{K}}(\omega) d\mu_{\mathbb{T}}(z).$$

Main results

Now, we are going to define a irreducible representation on $\mathbb{H}(G_\tau)$. With the above notations define $\pi : \mathbb{H}(G_\tau) \to U(L^2(\hat{K}))$ by

$$\pi(h,k,\omega,z)f(\xi) = \delta_H^{-1/2}(h) z \xi(k)\overline{\omega(k)} f((\xi\overline{\omega})_{h^{-1}}). \tag{1}$$

π is a continuous unitary representation of group $\mathbb{H}(G_\tau)$ to the Hilbert space $L^2(\hat{K})$. In the sequel, we show that π is irreducible when H is compact. Note that when H is a compact group, we normalize the Haar measure μ_H such that $\mu_H(H) = 1$.

Theorem

Let $\mathbb{H}(G_\tau) = (H \times_\tau K) \times_\theta (\hat{K} \times \mathbb{T})$ where H is a locally compact group and K is a locally compact abelian group. Then for φ, ψ in $L^2(\hat{K})$,

$$\int_{\mathbb{H}(G_\tau)} | \prec \varphi, \pi(h,k,\omega,z)\psi \succ |^2 d\mu_{\mathbb{H}(G_\tau)}(h,k,\omega,z) = \|\varphi\|_2^2 \|\psi\|_2^2. \qquad (2)$$

if and only if H is compact.

corollary

With notation as above, the representation π of $\mathbb{H}(G_\tau)$ on $L^2(\hat{K})$ is irreducible if H is compact.

example

Let K be an abelian locally compact group and $H = \{e\}$ (the trivial group). In this case the generalized weyl Heisenberg group $\mathbb{H}(G_\tau)$ coincides with the standard weyl Heisenberg group $G := K \times_\theta (\hat{K} \times \mathbb{T})$. In this case the irreducible representation of $G = K \times_\theta (\hat{K} \times \mathbb{T})$ on $L^2(\hat{K})$ is as follows:

$$\pi(k, \omega, z)f(\xi) = z\xi(k)\overline{\omega(k)}f(\xi\overline{\omega}). \quad (3)$$

Reminder

An irreducible representation π of $\mathbb{H}(G_\tau)$ on $L^2(\hat{K})$ is called square integrable if there exists a non zero element ψ in $L^2(\hat{K})$ such that

$$\prec \pi(.,.,.,.)\psi, f \succ \in L^2(\mathbb{H}(G_\tau)), \tag{4}$$

for all $f \in L^2(\hat{K})$.

Theorem

The representation π of the GWH group $\mathbb{H}(G_\tau) = (H \times_\tau K) \times_\theta (\hat{K} \times \mathbb{T})$ on $L^2(\hat{K})$ is square integrable if and only if H is compact.

Spectral Questions and Future Directions

(1) What is the spectrum of operators induced by π on $G.W.H groups$?

(2) How does the compactness of the group H influence the spectral properties of the representation π?

(3) How does the spectral behavior of generalized Weyl–Heisenberg groups differ from that of the classical Heisenberg group?

references

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