A ringed-space-like structure on coalgebras for noncommutative algebraic geometry

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Notation

k: a fixed algebraically closed field

Throughout this talk, algebras and schemes are defined over k.

Alg: the category of algebras and algebra homomorphisms

cAlg: the full subcategory of **Alg** whose objects are commutative algebra

Sch: the category of schemes and scheme morphisms

Two embeddings

The assignement $A \mapsto \mathrm{Spec}(A)$ induces a fully-faithful functor

$$\mathrm{Spec} : \mathbf{cAlg}^{op} \hookrightarrow \mathbf{Sch}$$

with a left adjoint

$$\Gamma : \mathbf{Sch} \to \mathbf{cAlg}^{op}$$

given by the global sections.

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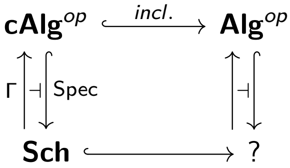
given by the global sections.

On the other hand, there is an inclusion functor

$$\mathbf{cAlg}^{op} \hookrightarrow \mathbf{Alg}^{op}.$$

Question

Can we extend the adjunction $\Gamma \dashv \mathrm{Spec}$ to $\mathbf{Alg}^{op}$(in a natural way)?



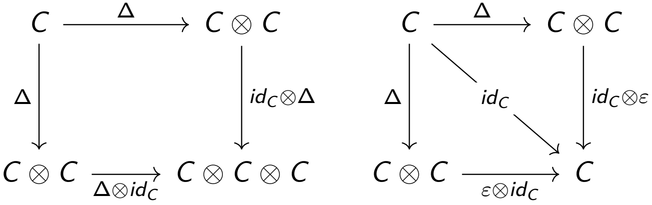
Coalgebras as noncommutative spectra

Reyes[2, 2025] discusses what an underlying structure of noncommutative spectrum should be and uses coalgebras.

Coalgebras

Definition (Coalgebras)

A triple (C, Δ, ε) is called a coalgebra if C is a vector space, $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ are linear maps that make the following commute:



A coalgebra homomorphism $f : C \to D$ is a linear map that is compatible with Δ's and ε's:

$$\forall x \in C \ \Delta_D(f(x)) = (f \otimes f)(\Delta_C(x)), \ \varepsilon_D(f(x)) = \varepsilon_C(x).$$

Cog denotes the category of coalgebras and coalgebra homomorphisms.

The finite dual coalgebras

Let A be an algebra and $A^* = \{\phi : A \to k | \phi \text{ is linear}\}$ be the dual space.

The subspace

$$A^\circ := \{\phi \in A^* | \exists I \subset \ker\phi,\ I \text{ is a (two-sided) ideal}, \dim A/I < \infty\}$$

has a natural coalgebra structure and is called the **finite dual coalgebra** of A.

Another description:

$$A^\circ = \varinjlim (A/I)^*$$

.

The assignment $A \mapsto A^\circ$ induces a functor $(-)^\circ : \mathbf{Alg}^{op} \to \mathbf{Cog}$.

The finite dual coalgebras: examples

If A is of finite dimension(as a vector space), then $A^\circ = A^*$.

For example, if $A = M_n(k)$, then the Δ and ε on $M^2(k) := M_2(k)^*$ are defined by

$$\Delta(e_{ij}) = \sum_{1 \leq k \leq n} e_{ik} \otimes e_{kj}, \ \varepsilon(e_{ij}) = \delta_{ij}$$

where e_{ij} is the matrix whose (i, j)-component is 1 and the other components are 0.

The finite duals of commutative algebras

Proposition

Let A be a commutative algebra. Then

$$A^\circ \simeq \bigoplus_{\mathfrak{p}} A^\circ_{\mathfrak{p}}$$

where $\mathfrak{p}$ runs over all the prime ideals such that $[\kappa(\mathfrak{p}) : k] < \infty$ where $\kappa(\mathfrak{p})$ is the residue field at $\mathfrak{p}$.

Since k is assumed to be algebraically closed, we have

$$A^\circ \simeq \bigoplus_{\mathfrak{p}} A^\circ_{\mathfrak{p}} = \bigoplus_{\mathfrak{M}} A^\circ_{\mathfrak{M}}$$

where $\mathfrak{M}$ runs over all the maximal ideals of codimension 1, i.e., $A/\mathfrak{M} \simeq k$.

The point-like elements of coalgebras

For a coalgebra C, the set of **point-like elements** is defined by

$$pts(C) = \{x \in C \mid \Delta(x) = x \otimes x,\ \varepsilon(x) = 1\}$$

and the construction induces a functor $pts : \mathbf{Cog} \to \mathbf{Set}$.

For example, we have

$$pts(A^\circ) = \mathbf{Alg}(A, k)$$

for all algebra A.

The finite dual and the maximum spectrum

The following diagram commutes:

$$\begin{array}{ccc}
\mathbf{Alg}^{op} & \xrightarrow{(-)^{\circ}} & \mathbf{Cog} \\
{\scriptstyle incl.}\uparrow & & \downarrow{\scriptstyle pts} \\
\mathbf{cAlg}^{op} & \xrightarrow[\mathbf{Alg}(-,k)]{} & \mathbf{Set}
\end{array}$$

In particular, if A is commutative and finitely generated as an algebra, then

$$\mathbf{Alg}(A, k) \simeq \mathrm{Max}(A) = |\,\mathrm{Spec}(A)|$$

where $\mathrm{Max}(-)$ stands for the maximal ideal spectrum and $|-|$ stands for the subset of closed points.

The Takeuchi underlying coalgebra

Let X be a scheme.

The Takeuchi coalgebra[5, 1974] $\mathbf{T}(X)$ of X is given by

$$\mathbf{T}(X) \simeq \bigoplus_{x \in \|X\|} \mathcal{O}^{\circ}_{X,x}$$

where

$$\|X\| = \{x \in X | [\kappa(x) : k] < \infty\}.$$

Here $\kappa(x)$ stands for the function field at x.

Another description(by Reyes):

$$\mathbf{T}(X) \simeq \varinjlim \Gamma(S, \mathcal{O}_S)^*$$

Properties of the Takeuchi coalgebra

The Takeuchi coalgebra induces a continuous(i.e., limit-preserving) functor $\mathbf{T} : \mathbf{Sch} \to \mathbf{Cog}$.

The following diagram commutes up to natural isomorphism:

$$\begin{array}{ccc} \mathbf{cAlg}^{op} & \xhookrightarrow{\;incl.\;} & \mathbf{Alg}^{op} \\ \mathrm{Spec}\big\downarrow & & \big\downarrow (-)^{\circ} \\ \mathbf{Sch} & \xrightarrow[\mathbf{T}]{} & \mathbf{Cog} \end{array}$$

Moreover, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{Sch}^{lf} & \xrightarrow{\quad \mathbf{T} \quad} & \mathbf{Cog} \\ & {\scriptstyle |-|}\searrow \quad \swarrow{\scriptstyle pts} & \\ & \mathbf{Set} & \end{array}$$

where $\mathbf{Sch}^{lf}$ is the full subcategory of $\mathbf{Sch}$ whose objects are locally of finite type.

Fully RFD algebras

If $A = k\langle x, y\rangle/(xy - yx - 1)$(the Weyl algebra), then $A^\circ = 0$.

We focus on algebras whose finite duals behave nicely:

Definition (Fully RFD algebras)

An algebra A is **fully residually finite dimensional(RFD)** if every finitely generated left A-module is a subdirect product of left A-modules of finite dimension.

Affine Noetherian PI algebras are fully RFD.

Commutative finitely generated algebras are fully RFD.

The finite duals give an adjunction

Let C be a coalgebra. The dual space $C^* = \{\phi : C \to k | \phi \text{ is linear}\}$ has a natural structure of algebra and the assignment $C \mapsto C^*$ defines a functor $(-)^* : \mathbf{Cog} \to \mathbf{Alg}^{op}$.

The functor $(-)^*$ is a left adjoint of the functor $(-)^\circ$:

$$\mathbf{Cog}(C, A^\circ) \simeq \mathbf{Alg}(A, C^*)$$

and there are natural transformations $A \to A^{\circ *}$ and $C \hookrightarrow C^{*\circ}$.

If A is fully RFD, then the natural homomorphism $A \to A^{\circ *}$ is injective. Every element in A is determined by how they act on each point of A° if we view A as an algebra of functions on A°.

The finite dual coalgebra+something

If A is a commutative algebra, then the corresponding affine scheme consists of

1. Spec(A): the set of prime ideals of A,
2. the Zariski topology on Spec(A), and
3. the structure sheaf on Spec(A).

Our aim is to equip the coalgebra A° with additional data when A is a fully RFD algebra.

Closed subsets of Spec(A)

Let A be a commutative finitely generated algebra.

Then every ideal $I \subset A$ defines a subset

$$Z(I) := \{\mathfrak{p} \in \mathrm{Spec}(A) | I \subset \mathfrak{p}\} \subset \mathrm{Spec}(A)$$

and the subsets of Spec(A) in this form are closed under the Zariski topology on Spec(A).

Closed subcoalgebras of A°

Let A be a fully RFD algebra.

Then every (two-sided) ideal $I \subset A$ defines a subset

$$Z^\circ(I) = \{\phi \in A^\circ | I \subset \ker \phi\} \subset A^\circ.$$

This is a subcoalgebra of A°, i.e., $\Delta(Z^\circ(I)) \subset Z^\circ(I) \otimes Z^\circ(I)$.

The subcoalgebras of A° in this form will be called **closed subcoalgebras**.

Subsets vs subcoalgebras

If X is a set, then the powerset of X forms a distributive lattice under $\cap$ and $\cup$:

$$(S_1 \cap S_2) \cup S_3 = (S_1 \cup S_3) \cap (S_2 \cup S_3),\ (S_1 \cup S_2) \cap S_3 = (S_1 \cap S_3) \cup (S_2 \cap S_3)$$

and

$$S_1 \cap \emptyset =, S_1 \cup \emptyset = S_1, S_1 \cap X = S_1, S_1 \cup X = X$$

for any subsets $S_1, S_2, S_3 \subset X$.

Subsets vs subcoalgebras

If C is a coalgebra, then the set of subcoalgebras forms a quantale under $\cap$ and $\vee$:

$$(D_1 \cap D_2) \vee D_3 = (D_1 \vee D_3) \cap (D_2 \vee D_3),\ (D_1 \vee D_2) \cap D_3 = (D_1 \cap D_3) \vee (D_2 \cap D_3)$$

and

$$D_1 \cap 0 = 0, D_1 \vee 0 = D_1, D_1 \cap C = D_1, D_1 \vee C = C$$

for any subcoalgebras $D_1, D_2, D_3 \subset C$ where

$$D_1 \vee D_2 := \Delta^{-1}(D_1 \otimes C + C \otimes D_2)$$

is the wedge product.

Furthermore,

$$pts(D_1 \vee D_2) = pts(D_1) \cup pts(D_2).$$

for any subcoalgebras $D_1, D_2 \subset C$.

Closed subsets of Spec(A)

Let A be commutative and finitely generated as an algebra.

The subsets $\emptyset = Z(A)$ and $\mathrm{Spec}(A) = Z(0)$ are closed.

The intersection of closed subsets of Spec(A) is closed:

$$\bigcap Z(I_i) = Z(\sum I_i).$$

The union of two closed subsets of Spec(A) is also closed:

$$Z(I_1) \cup Z(I_2) = Z(I_1 I_2).$$

Closed subspaces of A°

Let A be fully RFD.

The subspaces $0 = Z^\circ(A)$ and $A^\circ = Z^\circ(0)$ are closed.

The intersection of closed subcoalgebras of A° is a closed subcoalgebra:

$$\bigcap Z^\circ(I_i) = Z^\circ(\sum I_i).$$

The wedge product of two closed subcoalgebras of A° is also a closed subcoalgebra:

$$Z^\circ(I_1) \vee Z^\circ(I_2) = Z^\circ(I_1 I_2).$$

Corresponding notions

$$\text{a set } X = \mathrm{Spec}(A) \quad \rightsquigarrow \quad \text{a coalgebra } C = A^\circ$$

$$\text{a subset } Y \subset X \quad \rightsquigarrow \quad \text{a subcoalgebra } D \subset C$$

$$\text{a closed subset } Z(I) \subset X \quad \rightsquigarrow \quad \text{a closed subcoalgebra } Z^\circ(I) \subset C$$

$$\text{a sheaf } \mathcal{O} \text{ on } X \quad \rightsquigarrow \quad ???$$

“Open” subcoalgebras?

Let A be fully RFD and $I \subset A$ be an ideal.

We may define the “complement” of $Z^\circ(I)$ as

$$Z^\circ(I)^c := \bigcap_{C \cap Z^\circ(I) = 0} C \subset A^\circ \simeq \mathbf{T}(\mathrm{Spec}(A))$$

where C runs over all the subcoalgebras $C \subset A^\circ$ such that $C \cap Z^\circ(I) = 0$.

If A is commutative and finitely generated as an algebra, then

$$Z^\circ(I)^c \simeq \mathbf{T}(U)$$

where $U = \mathrm{Spec}(A) \backslash Z(I)$.

But it is not clear to me if this works functorially in noncommutative case....

a sheaf $\mathcal{O}$ on X $\rightsquigarrow$???

For the topological space $X = \text{Spec}(A)$, a sheaf $\mathcal{O}$ of algebras on X is a functor $\mathcal{O} : \mathcal{T}_X^{op} \to \textbf{Alg}$ where $\mathcal{T}_X$ is the partially ordered set of the **open subsets** of X with inclusion.

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For the coalgebra $C = A^\circ$, we do not define the notion of openess, and consider a fucntor $\mathcal{A} : \mathcal{P}_C^{op} \to \mathbf{Alg}$ where $\mathcal{P}_C$ is the partially ordered set of the **subcoalgebras** of C with inclusion.

Corresponding notions

If C is a coalgebra, then the dual space C^* has a natural algebra structure.

$$\text{a set } X = \mathrm{Spec}(A) \quad \rightsquigarrow \quad \text{a coalgebra } C = A^\circ$$

$$\text{a subset } Y \subset X \quad \rightsquigarrow \quad \text{a subcoalgebra } D \subset C$$

$$\text{a function } Y \to k \quad \rightsquigarrow \quad \text{a linear function } D \to k$$

$$\text{algebra } \mathbf{Set}(Y, k) \simeq k^Y \quad \rightsquigarrow \quad \text{algebra } D^*$$

We require $\mathcal{A}(D) \subset D^*$ for every subcoalgebra $D \subset C$.

Corresponding notions

$$\text{a set } X = \mathrm{Spec}(A) \quad \rightsquigarrow \quad \text{a coalgebra } C = A^\circ$$

$$\text{a subset } Y \subset X \quad \rightsquigarrow \quad \text{a subcoalgebra } D \subset C$$

$$\text{a closed subset } Z(I) \subset X \quad \rightsquigarrow \quad \text{a closed subcoalgebra } Z^\circ(I) \subset C$$

$$\text{a functor } \mathcal{O} : \mathcal{T}_X^{op} \to \mathbf{Alg} \quad \rightsquigarrow \quad \text{a functor } \mathcal{A} : \mathcal{P}_C^{op} \to \mathbf{Alg}$$

Construct a functor $\mathcal{A} : \mathcal{P}_C^{op} \to \mathbf{Alg}$

Let A be a fully RFD algebra and $C \subset A^\circ$ be a subcoalgebra.

For a subcoalgebra C, we define a collection $\mathfrak{F}_C$ of left ideals of A by

$$\mathfrak{F}_C := \{I \subset A | I \text{ is a left ideal, } Z^\circ(I) \cap C = 0\}.$$

The elements of $\mathfrak{F}_C$ are the left ideals of A whose vanishing sets are away from C.

The elements in those ideals are invertible on C.

Construct a functor $\mathcal{A} : \mathcal{P}_C^{op} \to \mathbf{Alg}$

We define a subset S_C of C^* by

$$S_C := \{x \in C^* | \exists I \in \mathfrak{F}_C \; a_C(I)x \subset a_C(A)\}$$

where a_C is the composition of the natural algebra homomorphisms

$$A \hookrightarrow A^{\circ *} \twoheadrightarrow C^*.$$

We denote by $\mathcal{A}(C)$ the subalgebra of C^* generated by S_C:

$$\mathcal{A}(C) := k\langle S_C \rangle \subset C^*.$$

Proposition (N.)

The construction of $\mathcal{A}(C)$ defines a functor $\mathcal{A}_A : \mathcal{P}_C^{op} \to \mathbf{Alg}$.

The ringed coalgebra A°

Let A be a fully RFD algebra.

We set

$$Q_A := \{Z^\circ(I) \mid I \subset A: \text{ two-sided ideal}\}$$

Now we may equip the coalgebra $C = A^\circ$ with the set Q of the closed subcoalgebras of A° and the functor $\mathcal{A}_A : \mathcal{P}_C^{op} \to \mathbf{Alg}$.

Ringed coalgebras

For a coalgebra C, $\mathcal{P}_C$ denotes the partially ordered set of the subcoalgebras of C with inclusion.

Definition (Ringed coalgebras)

A ringed coalgebra is a triple $(C, Q, \mathcal{A})$ where C is a coalgebra, Q is a collection of subcoalgebras of C closed under arbitrary $\bigcap$ and $\vee$, $\mathcal{A}$ is a subfunctor of $(-)^* : \mathcal{P}_C^{op} \to \mathbf{Alg}$

(i.e., it is a functor $\mathcal{A} : \mathcal{P}_C^{op} \to \mathbf{Alg}$ such that $\mathcal{A}(D) \subset D^*$ and the following diagram commutes for subcoalgebras $D_1 \subset D_2 \subset C$:

$$\begin{array}{ccc} D_2^* & \xrightarrow{\;i^*\;} & D_1^* \\ \uparrow & & \uparrow \\ \mathcal{A}(D_2) & \longrightarrow & \mathcal{A}(D_1) \end{array}$$

Here i is the inclusion $D_1 \hookrightarrow D_2$).

Morphisms of ringed spaces

Let $X_1 = (X_1, \mathcal{T}_1, \mathcal{O}_1)$ and $X_2 = (X_2, \mathcal{T}_2, \mathcal{O}_2)$ be ringed coalgebras.

A coalgebra homomorphism $f : X_1 \to X_2$ is said to be a morphism of ringed coalgebras if for any open subset $U \subset X_2$, we have

1. For all $U \in \mathcal{T}_2$, $f^{-1}(U) \in \mathcal{T}_1$, and
2. the map f induces natural algebra homomorphisms $\mathcal{O}_2(U) \to \mathcal{O}_1(f^{-1}(U))$ for all $U \in \mathcal{T}_2$.

Morphisms of ringed coalgebras

Let $\mathcal{C}_1 = (C_1, Q_1, \mathcal{A}_1)$ and $\mathcal{C}_2 = (C_2, Q_2, \mathcal{A}_2)$ be ringed coalgebras.

A coalgebra homomorphism $f : C_1 \to C_2$ is said to be a morphism of ringed coalgebras if for any subcoalgebra $D \subset C_2$, we have

1. For all $D \in Q_2$, $f^\dagger(D) \in Q_1$, and
2. the homomorphism f induces natural algebra homomorphisms $\mathcal{A}_2(D) \to \mathcal{A}_1(f^\dagger(D))$ for all $D \subset C_2$.

Here

$$f^\dagger(D) := \sum \{D' \subset C_1 : \text{subcoalgebra} \mid D' \subset f^{-1}(D)\}.$$

(The preimage $f^{-1}(D)$ is not a subcoalgebra in general.)

The category of ringed coalgebras will be denoted by **RC**.

The ringed coalgebra A°

$\mathbf{RFD_F}$: the full subcategory of **Alg** whose objects are fully RFD.

Theorem (N.)

Let A be a fully RFD algebra. The triple $(A^\circ, Q, \mathcal{A})$ is a ringed coalgebra. The assignement $A \mapsto A^\circ = (A^\circ, Q, \mathcal{A})$ defines a functor $(-)^\circ : \mathbf{RFD_F} \to \mathbf{RC}$.

Global sections

Let $(C, Q, \mathcal{A})$ be a ringed coalgebra. Then we define a functor $\Gamma : \mathbf{RC} \to \mathbf{Alg}$ by

$$\Gamma(C) := \mathcal{A}(C).$$

This functor will be called the global section functor.

Lemma (N.)

Let A be a fully RFD algebra and $I \subset A$ be an ideal. Then

$$\mathcal{A}_A(Z^\circ(I)) \simeq A/I.$$

Idea: If $C = Z^\circ(I) = (A/I)^\circ$, then $C^* \simeq (A/I)^{\circ *}$ and a_C is the composition

$$A \to A/I \hookrightarrow (A/I)^{\circ *}.$$

So $a_C(A) \simeq A/I$. We show that $S_C = a_C(A)$.

$$S_C := \{x \in C^* | \exists J \in \mathfrak{F}_C \; a_C(J)x \subset a_C(A)\}$$

For any left ideal $J \subset A$,

$$Z^\circ(J) \cap Z^\circ(I) = 0 \Leftrightarrow Z^\circ(I+J) = 0 \Leftrightarrow I + J = A.$$

Thus

$$a_C(I)x = a_C(I+J)x = a_C(A)x.$$

This implies $S_C = a_C(A) \simeq A/I$.

Corollary (N.)

Let A be a fully RFD algebra and $(A^\circ, Q_A, \mathcal{A}_A)$ be the associated ringed coalgebra. Then $\Gamma(A^\circ) = A$.

Condition (A) and an adjoint

Consider the following condition:
(A) The natural homomorphism

$$C \hookrightarrow C^{*\circ} \to \Gamma(C)^\circ$$

is a morphism from $(C, Q_C, \mathcal{A}_C)$ to $(\Gamma(C)^\circ, Q_{\Gamma(C)}, \mathcal{A}_{\Gamma(C)})$ in **RC**.

The ringed coalgebras that arise from fully RFD algebrassatisfy this condition.

Theorem

Let A be a fully RFD algebra and C be a ringed coalgebra satisfying (A). There is a natural bijective correspondence

$$\mathbf{RC}(C, A^\circ) \simeq \mathbf{RFD}(A, \Gamma(C))$$

given by $f \mapsto \Gamma(f)$.

Theorem

The functor $(-)^\circ : \mathbf{RFD_F} \to \mathbf{RC}$ given by the assignement $A \mapsto A^\circ = (A^\circ, Q, \mathcal{A})$ is fully-faithful.

Idea:

$$\mathbf{RC}(B^\circ, A^\circ) \simeq \mathbf{RFD_F}(A, \Gamma(B^\circ)) \simeq \mathbf{RFD_F}(A, B).$$

The ringed coalgebra A°

We have obtained a fully faithful functor $(-)^\circ : \mathbf{RFD_F} \hookrightarrow \mathbf{RC}$.

Next question: Can we turn schemes into ringed coalgebras? Can we make the following diagram commute?

$$\begin{array}{ccc} \mathbf{cAff}^{op} & \xhookrightarrow{incl.} & \mathbf{RFD}_{\mathbf{F}}^{op} \\ \Big\downarrow{\scriptstyle \mathrm{Spec}} & & \Big\downarrow{\scriptstyle (-)^\circ} \\ \mathbf{Sch} & \dashrightarrow_{?} & \mathbf{RC} \end{array}$$

Here **cAff** is the full subcategory of **Alg** whose objects are commutative finitely generated algebras.

Ringed coalgebra structure on $\mathbf{T}(X)$

We assume a scheme X to be locally of finite type. Then

$$\mathbf{T}(X) \simeq \bigoplus_{x \in |X|} \mathcal{O}^{\circ}_{X,x}$$

where $|X| \subset X$ is the set of closed points of X.

Every closed subscheme Y of X induces a injective coalgebra morphism $\mathbf{T}(Y) \hookrightarrow \mathbf{T}(X)$. In this way, $\mathbf{T}(Y)$ can be viewed as a subcoalgebra of $\mathbf{T}(X)$.

We define

$$Q_X := \{\mathbf{T}(Y) \subset \mathbf{T}(X) \mid Y \subset X \text{ is a closed subscheme}\}$$

Ringed coalgebra structure on $\mathbf{T}(X)$

We may define $\mathcal{A}_X$ to be the "largest" functor $P^{op}_{\mathbf{T}(X)} \to \mathbf{Alg}$ such that every scheme morphism $f : \mathrm{Spec}(A) \to X$ induces a ringed coalgebra morphism $A^\circ \to \mathbf{T}(X)$.

Theorem (N.)

The triple $(\mathbf{T}(X), Q, \mathcal{A})$ is indeed a ringed coalgebra and the assignement $X \mapsto \mathbf{T}(X) = (\mathbf{T}(X), Q_X, \mathcal{A}_X)$ defines a faithful functor $\mathbf{T} : \mathbf{Sch}^{lf} \hookrightarrow \mathbf{RC}$.

The ringed coalgebras that arise from schemes locally of finite type also satisfy condition (A):
(A) The natural homomorphism

$$C \hookrightarrow C^{*\circ} \to \Gamma(C)^\circ$$

is a morphism from $(C, Q_C, \mathcal{A}_C)$ to $(\Gamma(C)^\circ, Q_{\Gamma(C)}, \mathcal{A}_{\Gamma(C)})$ in $\mathbf{RC}$.

Commutative algebras give the same ringed coalgebras

Recall that there is a natrual isomorphism $\mathbf{T}(\mathrm{Spec}(A)) \simeq A^\circ$ of coalgebras for any commutative algebra A.

Theorem (N.)

Let A be commutative and finitely generated as an algebra and $X := \mathrm{Spec}(A)$. Under the identification $\mathbf{T}(\mathrm{Spec}(A)) \simeq A^\circ$, we have

$$\mathcal{A}_A(C) = \mathcal{A}_X(C)$$

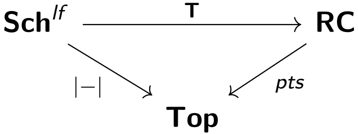
for any subcoalgebra $C \subset A^\circ$. In particular, $\mathbf{T}(\mathrm{Spec}(A)) \simeq A^\circ$ as ringed coalgebras.

The underlying topological space of a ringed coalgebra

If $C = (C, Q, \mathcal{A})$ is a ringed coalgebra, the set $pts(C)$ of point-like elements is endowed with the topology generated by subsets of the form $pts(D) = D \cap pts(C)$, $D \in Q$. This construction is functorial and defines $pts : \mathbf{RC} \to \mathbf{Top}$.

Theorem (N.)

The following diagram commutes:

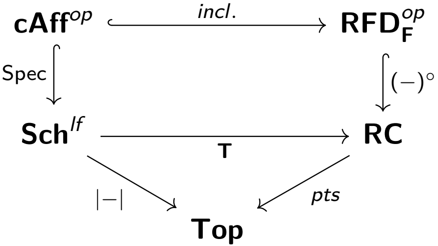


Proof: $|Z| = pts(\mathbf{T}(Z)) = \mathbf{T}(Z) \cap pts(\mathbf{T}(X)) = \mathbf{T}(Z) \cap |X|$.

Homebase

Theorem (N.)

The following diagram commutes up to natural isomorphism:



Here the round arrows stand for the fully faithful functors.

Fullness of $\mathbf{T}$

Proposition

Let A be a commutative finitely generated domain and let $X = \mathrm{Spec}(A)$. For every open subset $U \subset X$, we have a natural isomorphism

$$\mathcal{A}_X(\mathbf{T}(U)) \simeq \mathcal{O}_X(U).$$

Idea: We work on inside the algebra

$$\mathbf{T}(U)^* = \prod_{\mathfrak{m} \in |U|} \hat{A}_{\mathfrak{m}}.$$

Both $\mathcal{O}_X(U)$ and $\mathcal{A}_X(\mathbf{T}(U))$ can be viewed as subalgebras of $\mathbf{T}(U)^*$. Take an element $x \in \mathcal{O}_X(U)$ and finite affine covering $U = \bigcup_{1 \leq i \leq n} D(f_i)$ so that $x_{\mathfrak{m}}$ can be written as $\frac{a_i}{f_i^e}$ for all $\mathfrak{m} \in |D(f_i)|$. Then $I := (f_1^e, \cdots, f_n^e)$ satisfies $a_{\mathbf{T}(U)}(I)x \in a_{\mathbf{T}(U)}(A)$. This defines $\mathcal{O}_X(U) \hookrightarrow \mathcal{A}_X(\mathbf{T}(U))$ and show that it is also surjective.

Proposition

Let X be a integral scheme locally of finite type. For every open subscheme $U \subset X$(resp. closed subscheme $Z \subset X$), we have a natural isomorphism

$$\mathcal{A}_X(\mathbf{T}(U)) \simeq \mathcal{O}_X(U)(\text{resp. } \mathcal{A}_X(\mathbf{T}(Z)) \simeq \mathcal{O}_Z(Z)).$$

$\mathbf{IntSch}^{lf}$ denotes the full subcategory of $\mathbf{Sch}^{lf}$ whose objects are integral schemes locally of finte type.

Theorem

The functor $\mathbf{T} : \mathbf{IntSch}^{lf} \to \mathbf{RC}$ is fully-faithful.

Idea: It suffices to show the fullness of $\mathbf{T}$. Let $g : \mathbf{T}(X) \to \mathbf{T}(Y)$ be a morphism of ringed coalgebras. By applying *pts*, we obtain a continuous function $|X| \to |Y|$ which extends to a continous function $f : X \to Y$. Let $U \subset Y$ be an affine open subset. Then g restricts to a coalgebra homomorphism $\mathbf{T}(f^{-1}(U)) \to \mathbf{T}(U)$. We can show that this is a morphism of coalgebras and the global section functor gives $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$.

Homebase

Theorem (N.)

The following diagram commutes up to natural isomorphism:

$$\begin{array}{ccccc} \mathbf{Afd}^{op} & & \xhookrightarrow{\;\;incl.\;\;} & & \mathbf{RFD}_{\mathbf{F}}^{op} \\ {\scriptstyle \mathrm{Spec}}\big\downarrow & & & & \big\downarrow{\scriptstyle (-)^{\circ}} \\ \mathbf{IntSch}^{lf} & & \xhookrightarrow[\;\;\mathbf{T}\;\;]{} & & \mathbf{RC} \\ & {\scriptstyle |-|}\searrow & & \swarrow{\scriptstyle pts} & \\ & & \mathbf{Top} & & \end{array}$$

Here the round arrows stand for the fully faithful functors.

Modules over ringed coalgebras

Definition

Let $C = (C, Q_C, \mathcal{A}_C)$ be a ringed coalgebra. A *module* over C is a functor $\mathcal{F} : \mathcal{P}_C^{op} \to \mathbf{Vect}$ such that for every subcoalgebra $D \subset C$, $\mathcal{F}(D)$ is a left RFD module over $\mathcal{A}_C(D)$ and the following diagram commute:

$$\begin{array}{ccc} \mathcal{A}(D) \otimes \mathcal{F}(D) & \xrightarrow{\rho^D_{D'} \otimes \lambda^D_{D'}} & \mathcal{A}(D') \otimes \mathcal{F}(D') \\ \downarrow \mu_D & & \downarrow \mu_{D'} \\ \mathcal{F}(D) & \xrightarrow[\lambda^D_{D'}]{} & \mathcal{F}(D') \end{array}$$

where $\rho^D_{D'}$ and $\lambda^D_{D'}$ are restrictions defined in definition and $\mu_D, \mu_{D'}$ are the actions of the algebras over the modules.

Morphisms of modules over C are natural transformations respecting actions by $\mathcal{A}_C$. The modules over C form a category ${}_{A^\circ}\mathbf{Mod}$.

Modules over ringed coalgebra A°

Let A be a fully RFD algebra and M be a finitely generated left A-module.

Then the finite dual M° becomes a left A°-comodule. Furthermore, it produces a module $\hat{M}$ over A°. This defines a functor $(\hat{-}) : {}_A\mathbf{Mod}_{f.g.} \to {}_{A^\circ}\mathbf{Mod}$.

Comodules

Let C be a coalgebra. A left comodule $M = (M, \rho)$ over C is a pair of a vector space M and a linear map $\rho : M \to C \otimes M$ that makes the diagram on the left hand side commute. A comodule homomorphism $f : (M_1, \rho_1) \to (M_2, \rho_2)$ is a linear map $f : M_1 \to M_2$ that makes the diagram on the RHS commute.

$$\begin{array}{ccc} M & \xrightarrow{\rho} & C \otimes M \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle id\otimes\rho} \\ C \otimes M & \xrightarrow[\Delta\otimes id]{} & C \otimes C \otimes M \end{array} \qquad \begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ {\scriptstyle\rho_1}\downarrow & & \downarrow{\scriptstyle\rho_2} \\ C \otimes M_1 & \xrightarrow[id\otimes f]{} & C \otimes M_2 \end{array}$$

The category of left comodules over C will be denoted by ${}_C\mathbf{Mod}$.

Modules over ringed coalgebra A°

Let A be a fully RFD algebra and M be a finitely generated left A-module.

Then the finite dual M° becomes a left A°-comodule.

For every $C \subset A^\circ$, the cotensor product

$$C \Box M^\circ := \{x \in M^\circ | \rho(x) \in C \otimes M^\circ\}$$

is a left comodule over C.

Furthermore, it produces a module $\hat{M}$ over A°.

Theorem

The assignment $M \mapsto \hat{M}$ defines a fully-faithful functor $(\hat{-}) :_A \mathbf{Mod}_{f.g.} \to_{A^\circ} \mathbf{Mod}$.

Modules over ringed coalgebra $\mathbf{T}(X)$

Similarly, every coherent module $\mathcal{F}$ produces a module $\hat{\mathcal{F}}$ over $\mathbf{T}(X)$.

Theorem

The assignment $\mathcal{F} \mapsto \hat{\mathcal{F}}$ defines a fully-faithful functor $(\hat{-}) :_A \mathbf{Coh} \to_{\mathbf{T}(X)} \mathbf{Mod}$ if X is separated.

Future work

Question 1: Are the functors $A \mapsto (A^\circ, Q_A)$ and $X \mapsto (\mathbf{T}(X), Q_X)$ full? If so, can we construct $\mathcal{A}$ without mentioning ideals of A or closed subschemes of X?

Question 2: A commutative graded algebra A produces a projective scheme Proj A. Can we associate a ringed coalgebra to a noncommutative graded algebra A?

Question 3: If it is possible to associate a ringed coalgebra to a graded algebra, how is it related to Proj A as a Grothendieck category?

Question 4: If A and B are k-linearly Morita equivalent, what can we say about the ringed coalgebras A° and B°?

Related thoughts

The diagram that I want:

$$
\begin{array}{ccc}
\mathbf{cRing}^{op} & \xhookrightarrow{\ incl.\ } & \mathbf{Ring}^{op} \\
\Gamma \uparrow \dashv \downarrow Spec & & G \uparrow \dashv \downarrow S \\
\mathbf{Sch} & \xhookrightarrow{\ Q\ } & \mathcal{C}
\end{array}
$$

Using the category of modules

Let R be a ring. Then the category $\mathbf{Mod}R$ of right R-modules is a Grothendieck category. Every ring homomorphism $\varphi : R \to S$ induces an adjunction $(\varphi^*, \varphi_*) : \mathbf{Mod}S \to \mathbf{Mod}R$ and the natural isomorphism $1_R \otimes - : \varphi^*(R) \simeq S$.

Here

$$\varphi^*(M) = M \otimes_R S$$

and

$$\varphi_*(N) = \mathbf{Mod}S(S, N)$$

for all $M \in \mathbf{Mod}R$ and $N \in \mathbf{Mod}S$. The natural isomorphism $1_R \otimes -$ is given by $R \otimes_R S \xrightarrow{\sim} S$.

The right R-action on $\mathbf{Mod}S(S, N)$ is given by $(\phi \cdot r)(x) = \phi(\varphi(r)x)$ for all $\phi \in \mathbf{Mod}S(S, N)$, $r \in R$ and $x \in S$.

Using the category of modules

Let X be a quasi compact quasi separated scheme. Then the category $\mathbf{Qcoh}X$ of the quasi-coherent modules over X is a Grothendieck category. Every scheme morphism $f : X \to Y$ induces an adjunction $(f^*, f_*) : \mathbf{Qcoh}X \to \mathbf{Qcoh}Y$ and the natural isomorphism $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}- : f^*(\mathcal{O}_Y) \simeq \mathcal{O}_X$.

Definition

We mean by a quasi scheme a pair $(\mathcal{C}, \mathcal{O})$ of a Grothendieck category $\mathcal{C}$ and an object $\mathcal{O}$ of $\mathcal{C}$. A morphism of quasi schemes $(\mathcal{C}_1, \mathcal{O}_1) \to (\mathcal{C}_2, \mathcal{O}_2)$ is a triple (F, G, α) of adjoint functors $F : \mathcal{C}_2 \to \mathcal{C}_1$, $G : \mathcal{C}_1 \to \mathcal{C}_2$ and an isomorphism $\alpha : F\mathcal{O}_2 \xrightarrow{\sim} \mathcal{O}_1$. We denote by **qSch** the category of quasi schemes and morphisms.

A composition of $(F_1, G_1, \alpha_1) : (\mathcal{C}_1, \mathcal{O}_1) \to (\mathcal{C}_2, \mathcal{O}_2)$ and $(F_2, G_2, \alpha_2) : (\mathcal{C}_2, \mathcal{O}_2) \to (\mathcal{C}_3, \mathcal{O}_3)$ is given by $(F_1 F_2, G_2 G_1, \alpha_1 \circ F_1\alpha_2) : (\mathcal{C}_1, \mathcal{O}_1) \to (\mathcal{C}_3, \mathcal{O}_3)$.

The assignment $R \mapsto (\mathbf{Mod}R, R)$ gives a functor $\mathbf{Ring}^{op} \to \mathbf{qSch}$ which will be denoted by $\mathbf{Mod}_*$.

Likewise, the assignment $X \mapsto (\mathbf{Qcoh}X, \mathcal{O}_X)$ gives a functor $\mathbf{Sch} \to \mathbf{qSch}$ which will be denoted by Q. Note that if $X = \mathrm{Spec}(R)$ is affine, then $(\mathbf{Qcoh}X, \mathcal{O}_X)$ is naturally isomorphic to $(\mathbf{Mod}R, R)$ in $\mathbf{qSch}$.

Global section functor

Every morphism $(F, G, \alpha) : (\mathcal{C}_1, \mathcal{O}_1) \to (\mathcal{C}_2, \mathcal{O}_2)$ of quasi schemes naturally defines a ring homomorphism

$$\mathbf{End}(\mathcal{O}_2) \xrightarrow{F} \mathbf{End}(F(\mathcal{O}_2)) \overset{\alpha()\alpha^{-1}}{\simeq} \mathbf{End}(\mathcal{O}_1)$$

where the isomorphism $\mathbf{End}(F(\mathcal{O}_2)) \xrightarrow{\sim} \mathbf{End}(\mathcal{O}_1)$ is defined by sending every $\phi \in \mathbf{End}(F(\mathcal{O}_2))$ to $\alpha \circ \phi \circ \alpha^{-1} \in \mathbf{End}(\mathcal{O}_1)$.

Definition

We denote by $\mathbf{End} : \mathbf{qSch} \to \mathbf{Ring}^{op}$ the functor defined by sending every quasi scheme $\mathcal{C} = (\mathcal{C}, \mathcal{O})$ to the ring $\mathbf{End}(\mathcal{O})$. The functor **End** will be called the global section functor.

Proposition

Let $\mathcal{C} = (\mathcal{C}, \mathcal{O})$ be a quasi-scheme and R be a ring. Then there is an equivalence of categories

$$\mathbf{qSch}((\mathcal{C}, \mathcal{O}), (\mathbf{Mod}R, R)) \stackrel{\sim}{\to} \mathbf{Ring}(R, \mathbf{End}(\mathcal{O}))$$

given by sending (F, G, α) to the composite

$$R \simeq \mathbf{End}(R) \stackrel{F}{\to} \mathbf{End}(FR) \stackrel{\alpha()\alpha^{-1}}{\simeq} \mathbf{End}(\mathcal{O}).$$

Here, the isomorphism on the left sends every element $r \in R$ to the left multiplication by r.

Corollary

The functor $\mathbf{Mod}_*$ gives an equivalence of categories

$$\mathbf{Ring}(R, S) \xrightarrow{\sim} \mathbf{qSch}((\mathbf{Mod}S, S), (\mathbf{Mod}R, R))$$

given by $\varphi \mapsto \mathbf{Mod}_*(\varphi) = (\varphi^*, \varphi_*, 1_R \otimes -)$. Here, the set on the left-hand side is seen as a discrete category.

Related thoughts

$$
\begin{array}{ccc}
\mathbf{cRing}^{op} & \xhookrightarrow{\ incl.\ } & \mathbf{Ring}^{op} \\
\Gamma \uparrow \dashv \downarrow Spec & & \mathbf{End} \uparrow \dashv \downarrow \mathbf{Mod}_* \\
\mathbf{Sch} & \xrightarrow{\quad Q \quad} & \mathbf{qSch}
\end{array}
$$

Question: Is Q fully-faithful?

If $f : \mathbf{Qcoh} Y \to \mathbf{Qcoh} X$ respects the tensor structure, then it arizes from a scheme homomorphism $X \to Y$(by Brandenburg& Chrivasitu, [3, 2014]).

Thank you!(Reference)

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The finite dual coalgebras

The A° has the following description:

$$A^\circ = \{\phi \in A^* | \exists \phi_i, \psi_i \in A^* \ \forall a, b \in A \ \phi(ab) = \sum_i \phi_i(a)\psi_i(b) \ \}.$$

A° is a coalgebra togeather with $\Delta(\phi) := \sum_i \phi_i \otimes \psi_i$ and $\varepsilon(\phi) := \phi(1)$.

The original definition of Takeuchi underlying coalgebras

Let X be a scheme. The underlying coalgebra is a cocomutative coalgebra $\mathbf{T}(X)$ such that

$$\mathbf{Cog}(C, \mathbf{T}(X)) \simeq \mathbf{Sch}(\mathrm{Spec}(C^*), X)$$

for all cocomutative coalgebras C of finite dimension.

The coalgebra $\mathbf{T}(X)$ exists for every X and is given by

$$\mathbf{T}(X) \simeq \bigoplus_{x \in \|X\|} \mathcal{O}^{\circ}_{X,x}$$

where

$$\|X\| = \{x \in X | [\kappa(x) : k] < \infty\}.$$

Here $\kappa(x)$ stands for the function field at x.

Gabriel localization

Let A be a commutative domain and let $X = \mathrm{Spec}(A)$. For every open subset $U \subset X$, define

$$\mathfrak{F}_U := \{I \subset A \mid Z(I) \cap U = \emptyset\}.$$

Then the Gabriel localization

$$\varinjlim \mathbf{Mod}A(I, A)$$

of A in terms of $\mathfrak{F}_U$ is isomorphic to the section $\mathcal{O}_X(U)$.

Technical lemma

Let C_1, C_2, D_1 and D_2 be ringed coalgebras satisfying (A), f, g, h and i are coalgebra homomorphisms making the following diagram commute:

$$\begin{array}{ccc} C_1 & \xrightarrow{g} & C_2 \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle i} \\ D_1 & \xrightarrow[h]{} & D_2 \end{array}$$

If f and g are morphisms of ringed coalgebras and the composition

$$j : D_2 \xrightarrow{i} i(D_2) \hookrightarrow i(D_2)^{*\circ} \to \mathcal{A}_{C_2}(i(D_2))^{\circ}$$

of i and the natural coalgebra homomorphisms is an isomorphism of ringed coalgebras(hence i is injective), then h is a morphism of ringed coalgebras.

Sobrification

For a topological space X, we denote by S(X) the set of nonempty irreducible closed subsets of X. If T X is closed, then S(T) S(X). S(X) can be endowed with a topology where the closed subsets are of the form S(T) for some closed subset T X. S(X) is sober and this construction together with topological closure of images of continuous functions defines a functor from Top to the full subcategory Sob of Top whose objects are sober spaces. This topological space S(X) is known as the sobrification of X and the functor is the left adjoint to the inclusion functor from Sob to Top.

Sobrification

Lemma

Let X be a sober topological space such that the subset of closed points is dense. Then X is naturally isomorphic to $S(\|X\|)$.

Proof.

Since X is sober, the map $x \mapsto \overline{\{x\}}$ gives an isomorphism from X to $S(X)$. It is enough to show that the map $T \mapsto T \cap \|X\|$ is an isomorphism from $S(X)$ to $S(\|X\|)$. The map is bijective since the subset of closed points of X is dense. Note that a closed subset $T' \subset X$ is irreducible if and only if $T' \cap \|X\|$ is. Therefore $S(T) \subset S(X)$ for some closed $T \subset X$ correspond to $\{T' \cap \|X\| | T' \in S(T)\} = S(T \cap \|X\|) \subset S(\|X\|)$. □

pointed irreducible cocomutative subcoalgebras

We say a coalgebra is *pointed* if all simple subcoalgebras, i.e., the nonzero subcoalgebras that are minimal with respect to the containment, are of 1-dimensional. It is *irreducible* if any two nonzero subcoalgebras intersect nontrivially. A cocomutative coalgebras can always be written as a direct sum of its pointed irreducible subcoalgebras