Basic Definitions from Hartshorne's "Algebraic Geometry"

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1 Schemes

Definition 1 (Connected)**.** *A scheme X is* connected *if its underlying topological space is connected. A topological space is connected if for all nonempty open (closed) sets U and V such that $A \cap B = \emptyset$ and $A \cup B \neq X$.*

Proposition 2. *For a ring A, the following are equivalent:*

(1) The affine scheme $\text{Spec}(A)$ *is connected.*

(2) A contains no nontrivial idempotents. That is, if $e^2 = e$, then $e = 0, 1$.

(3) A is not isomorphic to a product of rings $A_1 \times A_2$.

Proof. (1) $\iff$ (2) If A has some idempotent $e \neq 0, 1$, then the sets $V(e)$ and $V(1-e)$ are nonempty, disjoint, and their union is all of $\text{Spec}(A)$. Indeed, e and $1-e$ are not units since they are zero divisors so $V(e), V(1-e) \neq \emptyset$ and using that $e^2 = e$ we have

$$V(e) \cup V(1-e) = V(e(1-e)) = V(0) = \text{Spec}(A)$$
$$V(e) \cap V(1-e) = V((e) + (1-e)) = V((1)) = \emptyset.$$

Therefore, $\text{Spec}(A)$ is disconnected.

Conversely, if $\text{Spec}(A)$ is disconnected, there are ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $V(\mathfrak{a}) \cup V(\mathfrak{b}) = \text{Spec}(A)$ and $V(\mathfrak{a}) \cap V(\mathfrak{b}) = \emptyset$. This is equivalent to the statement that $\mathfrak{a} + \mathfrak{b} = (1)$ and $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{N}_A$, where $\mathfrak{N}_A$ is the nilradical of A. This implies there exists $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ such that $x + y = 1$. Multiply through by x and subtract to get that $x^2 - x = xy \in \mathfrak{a} \cap \mathfrak{b}$. That is, $x(x-1)$ is nilpotent. For some n, we then have $x^n(x-1)^n = 0$. Since x^n and $(x-1)^n$ are coprime, from the Chinese remainder theorem, we have that $A \simeq A/(x^n) \times A/((x-1)^n)$. The preimages of $(1, 0)$ and $(0, 1)$ from this map are nontrivial idempotents in A.

(2) $\iff$ (3) If A has a nontrivial idempotent e, then (e) and $(1-e)$ are ideals such that $(e) + (1-e) = (1)$, $(e)(1-e) = (e) \cap (1-e) = (0)$. From the Chinese remainder theorem, this implies that the natural map $A \mapsto A/(e) \times A/(1-e)$ is an isomorphism.

Conversely, if $\varphi : A_1 \times A_2 \mapsto A$ is an isomorphism, $\varphi(1, 0)$ and $\varphi(0, 1)$ are nontrivial idempotents of A. □

Example 3. *Any local ring $(A, \mathfrak{m})$ has no nontrivial idempotents and so the affine scheme Spec(A) is connected. To see this, let $e \in A$ be such that $e(1-e) = 0$. If e is a unit, then multiplying this equation by e^{-1} implies that $e = 1$. If e is not a unit, then $e \in \mathfrak{m}$ so that $1 - e \notin \mathfrak{m}$. That is, $1 - e$ is a unit. Multiplying the equation from before by $(1-e)^{-1}$, we have that $e = 0$. Therefore, $e = 0$ or $e = 1$ as desired.*

Example 4. *Projective n-space, $\mathbb{P}^n$ is connected since it is the union of $n + 1$ affine schemes that are connected and their intersection is nonempty.*

Definition 5 (Irreducible)**.** *A scheme is* irreducible *if its underlying topological space is irreducible. A topological space X is irreducible if any of the following equivalent conditions hold:*

1. *For every pair of nonempty open sets U and V, $U \cap V \neq \emptyset$.*
2. *X cannot be written as the union of two proper, closed sets.*
3. *Every nonempty open set is dense in X.*

Proposition 6. *An affine scheme Spec(A) is irreducible if and only if the nilradical $\mathfrak{N}_A$ is prime.*

Proof. If the nilradical $\mathfrak{N}_A$ is prime, let $U = \text{Spec}(A) \setminus V(\mathfrak{a})$ be a nonempty open set. Since U is nonempty, $V(\mathfrak{a}) \neq \text{Spec}(A)$. In particular, there is some prime ideal $\mathfrak{p}$ such that $\mathfrak{a} \not\subseteq \mathfrak{p}$. This implies $\mathfrak{a} \not\subseteq \mathfrak{N}_A$ so that $\mathfrak{N}_A \in U$. Therefore, every pair of nonempty open sets U and V, $\mathfrak{N}_A \in U \cap V$ and so Spec(A) is irreducible.

If the nilradical $\mathfrak{N}_A$ is not prime, there exists $a, b \in A$ such that $ab \in \mathfrak{N}_A$, but $a, b \notin \mathfrak{N}_A$. This implies there are prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $a \notin \mathfrak{p}$ and $b \notin \mathfrak{q}$. That is, $V(a)$ and $V(b)$ are proper closed sets. We have the following since $ab \in \mathfrak{N}_A$.

$$V(a) \cup V(b) = V(ab) = \text{Spec}(A)$$

Therefore, Spec(A) is not irreducible. □

Definition 7 (Reduced)**.** *A scheme $(X, \mathcal{O})$ is* reduced *if for every open set U, $\mathcal{O}(U)$ has no nilpotents. Equivalently, a scheme $(X, \mathcal{O})$ is reduced if for every $\mathfrak{p} \in X$, $\mathcal{O}_\mathfrak{p}$ has no nilpotents.*

Proposition 8. *An affine scheme Spec(A) is reduced if and only if $\mathfrak{N}_A = 0$.*

Proof. If Spec(A) is reduced, since Spec(A) is open, $\mathcal{O}(\text{Spec}(A)) = A$ should have no nilpotents. That is, $\mathfrak{N}_A = 0$.

Conversely, if $\mathfrak{N}_A = 0$, then all localizations of A have no nilpotents either. This implies each $A_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$ has no nilpotents and that Spec(A) is reduced.

(The hard part of this, which is not included is showing the equivalence in the definition.) □

Definition 9 (Integral)**.** *A scheme $(X, \mathcal{O})$ is* integral *if for every open set U, $\mathcal{O}(U)$ is an integral domain. Equivalently, $(X, \mathcal{O})$ is integral if it is reduced and irreducible.*

Proposition 10. *An affine scheme Spec(A) is integral if and only if A is an integral domain.*

Proof. If Spec(A) is integral, then $\mathcal{O}(\text{Spec}(A)) = A$ is an integral domain.

Conversely, if A is an integral domain, $\mathfrak{N}_A = 0$ and is prime so Spec(A) is reduced and irreducible.

(Again, the hard part is in the equivalence included in the definition.) □

Definition 11 (Locally Noetherian)**.** *A scheme $(X, \mathcal{O})$ is* locally Noetherian *if it can be covered by affine schemes Spec(A_i), where each A_i is Noetherian.*

Example 12. *Projective n-space, $\mathbb{P}^n$, can be covered by $n+1$ affine schemes in the usual way and so is locally Noetherian.*

Example 13. *Any toric variety constructed by a fan is locally Noetherian by definition.*

Definition 14 (Noetherian)**.** *A scheme X is* Noetherian *if it is locally Noetherian and quasi-compact. Equivalently, X is Noetherian if it can be covered by finitely many open affine schemes $Spec(A_i)$ where each A_i is Noetherian.*

Example 15. *Any affine scheme defined from a Noetherian ring.*

Example 16. *Again, projective n-space is locally Noetherian and quasi-compact. Therefore, $\mathbb{P}^n$ is Noetherian.*

Definition 17 (Noetherian Space)**.** *A topological space X is a* Noetherian space *if any of the following equivalent conditions hold:*

1. *Every ascending chain of open sets is eventually constant.*
2. *Every descending chain of closed sets is eventually constant.*

Example 18. *If A is a Noetherian ring, then $Spec(A)$ is a Noetherian space. This follows since every descending chain of closed sets corresponds uniquely to an increasing chain of radical ideals in A (and vice versa). That is, every descending chain of closed sets in $Spec(A)$ is eventually constant.*

Definition 19 (Open Subscheme)**.** *An* open subscheme *of a scheme X is a scheme U whose topological space is an open subset of X, and whose structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_X|_U$ of the structure sheaf of X.*

Example 20. *Let $(X, \mathcal{O}_X)$ be a scheme, and let U be an open subset of X. Then $(U, \mathcal{O}_X|_U)$ is a scheme (an open subscheme of X). We call this the* induced scheme structure *on the open set U.*

Definition 21 (Closed Subscheme)**.** *A* closed subscheme *of a scheme X is an equivalence class of closed immersions, where we say that the morphisms of schemes $f : Y \longrightarrow X$ and $f' : Y' \longrightarrow X$ are equivalent if there is an isomorphism $i : Y' \longrightarrow Y$ such that $f' = f \circ i$.*

Definition 22 (Dimension)**.** *The* dimension *of a scheme is the dimension of its underlying topological space. The dimension of a topological space is the length of a maximal ascending chain $U_0 \subset U_1 \subset \ldots \subset U_n$ of irreducible subsets.*

2 Morphisms of schemes

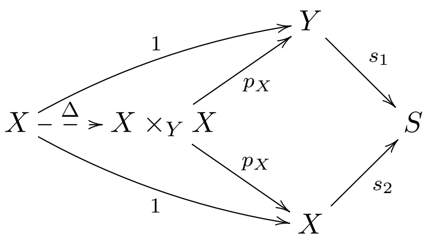
Definition 23 (Open Immersion)**.** *An* open immersion *is a morphism of schemes $f : X \longrightarrow Y$ which induces an isomorphism of X with an open susbcheme of Y.*

Definition 24 (Closed Immersion). *A* closed immersion *is a morphism of schemes $f : X \longrightarrow Y$ such that f induces a homeomorphism on the topological spaces, sending X to a closed subset of Y, and moreover the induced map of sheaves $f^{\#} : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective.*

Example 25. *Let A be a ring and I an ideal of A. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(A/I)$. Then the ring homomorphism $\pi : A \longrightarrow A/I$ induces a morphism of schemes $\operatorname{spec}(\pi) : Y \longrightarrow X$ which is a closed immersion. The map $\operatorname{spec}(\pi)$ is a homeomorphism of Y onto the closed subset $V(I)$ of X, and the map of structure sheaves $\operatorname{spec}(\pi)^{\#} : \mathcal{O}_X \longrightarrow \operatorname{spec}(\pi)_*\mathrm{O}_Y$ is surjective because it is surjective on the stalks.*

Definition 26 (Separated Morphism). *Let $f : X \to Y$ be a morphism of schemes. The* diagonal morphism *is the unique morphism $\Delta : X \to X \times_Y X$ whose composition with both projection maps $p_1, p_2 : X \times_Y X \to X$ is the identity map of $X \to X$. We say that the morphism f is* separated *if the diagonal morphism Δ is a closed immersion. In that case we say X is* separated over Y*. A scheme X is separated if it is separated over* $\operatorname{Spec}\mathbb{Z}$*.*

We can see the diagonal morphism as:



where note that all triangles and squares commute. Moreover, we have:

$$f \text{ is separated over } Y \iff \Delta \text{ is closed immersion.} \implies X \text{ is separated over } Y,$$

$$X \text{ is separated} \iff X \text{ is separated over } \operatorname{Spec}\mathbb{Z}.$$

Remark 27. *Let k be a field, let $X_1 = X_2 = \mathbb{A}^1_k = \operatorname{Spec}(k[x])$, let P be a point in $\mathbb{A}^1_k$ corresponding to the maximal ideal (x). Let $U_1 = U_2 = \mathbb{A}^1_k - \{(0)\}$, and $\psi : U_1 \to U_2$ is just the identification map (which induces an isomorphism of locally ringed space between them). Then we can glue X_1 and X_2 via ψ, so we have X the gluing of X_1 and X_2. In this case, Δ is not a closed immersion because $\Delta(sp(X_1)) = \Delta(sp(X_2))$ is a set containing one generic point, so its closure is the whole space, containing other generic points, and thus it is not closed in $sp(X)$.*

Proposition 28. *If $f : X \to Y$ is any morphism of affine schemes, then f is separated.*

Corollary 29. *A morphism $f : X \to Y$ is separated iff $\Delta(X)$ is a closed subset of $X \times_Y X$.*

Theorem 30. *Let $f : X \to Y$ be a morphism of schemes, X Noetherian. Then f is separated if and only if the following holds. For all k field, and all R valuative ring with quotient field k, $T := \mathrm{Spec}(R), U = \mathrm{Spec}(k)$, $i : U \to T$ induced by the inclusion $R \subseteq k$ and given maps making commutative the diagram:*

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ T & \longrightarrow & Y \end{array}$$

then there exists at most one morphism $T \to X$ making the following diagram commutative:

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ T & \longrightarrow & Y \end{array}$$

Corollary 31. *We have that:*

1. *Open and closed immersions are separated.*
2. *Composition of separated morphisms is separated.*
3. *Product of separated morphisms is separated.*
4. *If $g \circ f$ is a separated morphism, then f is separated.*
5. *A morphism $f : X \to Y$ is separated if and only if Y is covered by open subsets $\{U_i\}_{i \in I}$ such that $f^{-1}(U_i) \to V_i$ is separated for all $i \in I$.*

Definition 32 (Proper morphism)**.** *A morphism of schemes $f : X \to Y$ is proper if all the following hold:*

1. *f is separated,*
2. *f is of finite type,*
3. *f is universally closed, that is:*
   (a) *closed means image of any closed subsets is closed,*
   (b) *universally closed means it is closed and for any scheme morphism $Y' \to Y$ the corresponding morphism $f' : X' \to Y'$ obtained by base extension is also closed, where $X' = X \times_Y Y'$.*

Remark 33. *Let k be a field, X be an affine line over k, that is, $X = \mathbb{A}^1_k$, then $f : X \to k$ is separated and of finite type but not proper. Since $X \times_k X \to X$ is not closed: we send $V(xy - 1)$ to $X - \{0\}$.*

Theorem 34. *Let $f : X \to Y$ be a morphism of finite type, X Noetherian. Then f is proper if and only if the following holds. For all k field and $\forall R$ valuative ring with quotient field k, $T := \mathrm{Spec}(R), U = \mathrm{Spec}(k)$, $i : U \to T$ induced by the inclusion $R \subseteq k$ and given maps making the following diagram commutative:*

$$\begin{CD} U @>>> X \\ @V{i}VV @VV{f}V \\ T @>>> Y \end{CD}$$

then there exists at most one morphism $T \to X$ making the following diagram commutative:

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ T & \longrightarrow & Y \end{array}$$

Note that in the last two Theorems we just changed $f : X \to Y$ from being any morphism of schemes to being any morphism of schemes of finite type, and changed being separated to being proper.

Corollary 35. *We have that:*

1. *Cosed immersions are proper.*
2. *Composition of proper morphisms is proper.*
3. *Product of proper morphisms is proper.*
4. *If $g \circ f$ is a proper morphism, then f is proper.*
5. *Any projection morphism is proper.*

Definition 36 (Projective n-space)**.** *Let Y be a scheme.*

1. *The* projective n-space over Y, *denoted by $\mathbb{P}^n_Y$, is $\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\mathbb{Z}} Y$.*
2. *A morphism $f : X \to Y$ is projective if and only if $f = i \circ \pi$ where $i : X \to \mathbb{P}^n_Y$ is a closed immersion and $\pi : \mathbb{P}^n_Y \to Y$ is a projection.*
3. *A morphism $f : X \to Y$ is quasi-projective if and only if $f = j \circ \pi$ where $j : X \to X'$ is open immersion and $\pi : \mathbb{P}^n_Y \to Y$ is projective morphism.*

Example 37. *Let A be a ring, S a graded ring with $S_0 = A$ such that S is finitely generated as an A-algebra by S_1. Then $\mathrm{Proj}(S) \to \mathrm{Spec}(A)$ is a projective morphism.*

Notice that since S is a quotient of $A[x_0, \cdots, x_n]$, then $A[x_0, \cdots, x_n] \to S$ gives $\mathrm{Proj}(S) \to \mathrm{Proj}(A[x_0, \cdots, x_n]) = \mathbb{P}^n_A$, and $A \to A[x_0, \cdots, x_n]$ gives $\mathrm{Proj}(A[x_0, \cdots, x_n]) \to \mathrm{Spec}(A)$.

Theorem 38. *Projective morphisms of Noetherian schemes is proper. Quasi-projective morphism of Noetherian schemes is of finite type and separated.*

Proposition 39. *Let k be an algebraically closed field, and t a functor from the category of varieties to the category of schemes (notice that t is fully faithful). Then, the image of t is the set of quasi-projective integral schemes over k. Moreover, the image of the set of projective varieties is the set of projective integral schemes, and the image of any variety V is an integral separted scheme of finite type over k.*

Definition 40. *An* abstract variety *is an integral separated scheme of finite type over an algebraically closed field k. A* complete abstract variety *is an abstract variety which is proper over k.*

Remark 41. *A modern counterexample of an abstract variety that is not in the image of t may be constructed via a non-quasiprojective variety that is generated by a toric variety. However, this is not completely clear to the authors since this may be a non-normal toric variety.*

3 Constructions

Definition 42 (S-scheme)**.** *Let S be a fixed scheme. A* scheme over S, *i.e. an S-scheme, is a scheme X together with a morphism $X \to S$*

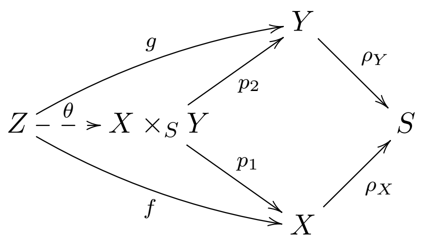
Example 43. *A vector bundle E over a scheme S with a map $E \to S$ is an S-scheme.*

Definition 44 (S-morphism)**.** *Let X and Y be S-schemes with respective maps $p : X \to S$ and $q : Y \to S$. Then an S-*morphism *is a morphism of schemes $f : X \to Y$ such that $p = q \circ f$*

Example 45. *Let S be a scheme and X with $X \to S$ an S-scheme. Viewing S as an S-scheme with $id : S \to S$, the S-morphism mapping $S \to X$ is called an S-section.*

Proposition 46. *Let k be an algebraically closed field. There is a natural fully faithful functor from the category of varieties over k to schemes over k.*

Definition 47 (Fibered Product)**.** *Let S be a scheme, and let $(X, \rho_X), (Y, \rho_Y)$ be S-schemes. The* fibered product *of X and Y over S, denoted $X \times_S Y$, is a scheme together with projection morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ such that $\rho_X \circ p_1 = \rho_Y \circ p_2$ and also for any given S-scheme Z with given morphism $f : Z \to X$ and $g : Z \to Y$ such that $\rho_X \circ f = \rho_Y \circ g$, then there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$ and $g = p_2 \circ \theta$. i.e. the following diagram commutes for any such Z:*



Example 48. *Let $X = Spec(A)$, $Y = Spec(B)$ and $S = Spec(R)$ be affine schemes where X and Y are S-schemes. Then A and B are R-algebras and the fibered product is $X \times_S Y = Spec(A \otimes_R B)$, up to unique isomorphism.*

Definition 49 (Fiber of a Morphism)**.** *Let $f : X \to Y$ be a morphism of schemes and let $y \in Y$ be a point. Let $k(y)$ be the residue field of y, and let $Spec(k(y)) \to Y$ be the inclusion map. Then the* fiber of the morphism f over the point y *is the scheme $X_f = X \times_Y Spec(k(y))$.*

Example 50. *Let k be an algebraically closed field. Suppose $X = Spec(k[x, y, t]/(xy-t))$, $Y = Spec(k[t])$, and $f : X \to Y$ is the morphism determined by the natural homomorphism $k[t] \to k[x, y, t]/(ty - x^2)$. Then for $a \in k$, X_a is the irreducible hyperbola $xy = a$ when $a \neq 0$. If $a = 0$, X_0 is the union of the x-axis and y-axis.*