6.3. Systems of linear differential equations.

A <u>differential equation</u> is an equation involving functions and their derivatives. There are many tools to solve them, such as separation of variables, variations of parameters, and using ansatz, just to mention a few.

<u>Example:</u> Solve the differential equation: $x'(t) = k\,x(t)$.

We rewrite the equation as $\frac{x'(t)}{x(t)} = k$ and integrate each side of the equality:

$$\log(x(t)) = \int \frac{x'(t)}{x(t)}\,dt = \int k\,dt = kt$$

using the Fundamental Theorem of Calculus. Then adding the constant we have:

$$\log(x(t)) = kt + D \quad \text{so} \quad x(t) = e^{\log(x(t))} = e^{kt+D} = e^{kt}\cdot e^{D}.$$

Since $x(0) = e^{D}$, this means: $x(t) = x(0)\cdot e^{kt}$. This is the general solution.

<u>Example:</u> Give two linearly independent solutions to: $\frac{\partial^2 u(x,t)}{\partial t^2} = k^2 \frac{\partial^2 u(x,t)}{\partial x^2}$, the wave equation.

Let's set $k=1$ at first for simplicity. The equation is telling us that we are looking for a function on two variables such that deriving it twice gives the same result regardless of which variable we are deriving. We know three functions that behave like themselves when derived twice: exponentials, sine, and cosine. We also know that sine and cosine are not linearly independent with the exponential when the scalars are the complex numbers: $e^{ix} = \cos(x) + i \cdot \sin(x)$ for x a real number.

Let's stick with just sine and cosine, and since we have to write them as functions in the variables x and t, let's just add them up for simplicity. Our ansatz are $\sin(x+t)$ and $\cos(x+t)$, giving:

$$\frac{\partial^2 \sin(x+t)}{\partial t^2} = -\sin(x+t) = \frac{\partial^2 \sin(x+t)}{\partial x^2} \quad \text{and}$$

$$\frac{\partial^2 \cos(x+t)}{\partial t^2} = -\cos(x+t) = \frac{\partial^2 \cos(x+t)}{\partial x^2}.$$

To incorporate a general k, we can do it multiplying the variable t, since we know it will come outside when taking derivatives. Our ansatz are then $\sin(x+kt)$ and $\cos(x+kt)$, giving:

$$\frac{\partial^2 \sin(x+kt)}{\partial t^2} = -k^2 \sin(x+kt) = k^2 \frac{\partial^2 \sin(x+kt)}{\partial x^2} \quad \text{and}$$

$$\frac{\partial^2 \cos(x+kt)}{\partial t^2} = -k^2 \cos(x+kt) = k^2 \frac{\partial^2 \cos(x+kt)}{\partial x^2}.$$

To keep sine and cosine linearly independent, we add and subtract the constant k independently. A very general solution is:

$$x(t) = a \cdot \sin(x+kt) + b \cdot \cos(x-kt) \quad \text{for } a \text{ and } b \text{ real scalars.}$$

A <u>system of differential equations</u> is a collection of equations involving functions and their derivatives. The main tool we will use to solve them is diagonalization. In two dimensions a system of differential equations looks like:

$$\begin{cases} y_1'(t) = a_{11}\, y_1(t) + a_{12}\, y_2(t) \\ y_2'(t) = a_{21}\, y_1(t) + a_{22}\, y_2(t) \end{cases} \quad \text{namely} \quad \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

In general, a system of differential equations looks like $\vec{y}'(t) = A\vec{y}(t)$ where A is an $n \times n$ matrix of scalars, $\vec{y}(t)$ is an $n\times 1$ vector of functions, and $\vec{y}'(t)$ is an $n \times 1$ vector with the derivatives of the aforementioned functions, in order. To solve such a system we may be tempted to generalize the case 1×1 we saw above, and use the function $\vec{y}(t) = \vec{y}(0) \cdot e^{At}$ where we defined the matrix exponential:

$$e^{At} = Id_n + At + \frac{(At)^2}{2} + \frac{(At)^3}{3} + \dots = \sum_{i=0}^{\infty} \frac{(At)^i}{i!}.$$

With this definition we indeed have $\vec{y}'(t) = \vec{y}(0) A e^{At} = A\vec{y}(t)$, but we have introduced the complexity of having to compute e^{At}. We will do this shortly, but before let's try to use a function like $\vec{y}(t) = e^{\lambda t}\vec{v}$ for $\vec{v}$ a fixed vector in $\mathbb{R}^n$. We have:

$$\vec{y}'(t) = \lambda e^{\lambda t}\vec{v} = \lambda \vec{y}(t)$$

so we will have $\vec{y}'(t) = A\vec{y}(t)$ if:

$$e^{\lambda t}\lambda \vec{v} = \vec{y}'(t) = A\vec{y}(t) = A e^{\lambda t}\vec{v} = e^{\lambda t} A\vec{v} \quad \text{namely} \quad \lambda\vec{v} = A\vec{v}.$$

Thus if $\vec{v}$ is an eigenvector of A of eigenvalue λ, then $\vec{y}(t) = e^{\lambda t}\vec{v}$ is a solution of $\vec{y}'(t) = A\vec{y}(t)$. For each eigenvector of A we have a solution of $\vec{y}'(t) = A\vec{y}(t)$.

<u>Example:</u> Solve the system of differential equations:

$$\begin{cases} y_1'(t) = -2y_1(t) + y_2(t) \\ y_2'(t) = y_1(t) - 2y_2(t) \end{cases} \quad \text{with } \vec{y}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \text{ and find its stable state.}$$

In matrix form, the system is:

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad \text{say} \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

To find solutions of $\vec{y}'(t) = A\vec{y}(t)$ we find the eigenvectors and eigenvalues of A:

$$\det(A - \lambda\cdot Id_2) = \det\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} = \lambda^2 + 4 + 4\lambda - 1 = \lambda^2 + 4\lambda + 3 = (\lambda+1)(\lambda+3).$$

$$\lambda = \frac{-4 \pm \sqrt{16 - 4\cdot 1\cdot 3}}{2\cdot 1} = \begin{cases} \frac{-4+2}{2} = -1. \\ \frac{-4-2}{2} = -3. \end{cases}$$

For $\lambda_1 = -1$:

$$\vec{0} = (A - \lambda_1 Id_2)\vec{x} = \begin{bmatrix} -2+1 & 1 \\ 1 & -2+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{so} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

because $c_1 + c_2 = \vec{0}$.

For $\lambda_2 = -3$:

$$\vec{0} = (A - \lambda_2 Id_2)\vec{x} = \begin{bmatrix} -2+3 & 1 \\ 1 & -2+3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{so} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

because $c_1 - c_2 = \vec{0}$.

So $e^{\lambda_1 t}\vec{v}_1 = \begin{bmatrix} \frac{1}{e^t} \\ \frac{1}{e^t} \end{bmatrix}$ and $e^{\lambda_2 t}\vec{v}_2 = \begin{bmatrix} \frac{1}{e^{3t}} \\ \frac{-1}{e^{3t}} \end{bmatrix}$ are linearly independent solutions. Since the system has dimension two, these form a basis of all solutions, so a general solution is a linear combination of them: $\vec{y}(t) = c_1 \cdot \begin{bmatrix} \frac{1}{e^t} \\ \frac{1}{e^t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{e^{3t}} \\ \frac{-1}{e^{3t}} \end{bmatrix}$ for some scalars c_1 and c_2. To find the unique solution satisfying $\vec{y}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ we solve:

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \vec{y}(0) = c_1 \cdot \begin{bmatrix} \frac{1}{e^0} \\ \frac{1}{e^0} \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{e^{3\cdot 0}} \\ \frac{-1}{e^{3\cdot 0}} \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$ with augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & -1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{array}\right] \quad \text{so} \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

and the solution is $\vec{y}(t) = \frac{1}{2} \cdot \begin{bmatrix} \frac{1}{e^t} \\ \frac{1}{e^t} \end{bmatrix} + \frac{3}{2} \cdot \begin{bmatrix} \frac{1}{e^{3t}} \\ \frac{-1}{e^{3t}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2e^t} + \frac{3}{2e^{3t}} \\ \frac{1}{2e^t} - \frac{3}{2e^{3t}} \end{bmatrix}$. The stable state occurs when $t \to \infty$, namely:

$$\lim_{t\to\infty} \vec{y}(t) = \lim_{t\to\infty} \begin{bmatrix} \frac{1}{2e^t} + \frac{3}{2e^{3t}} \\ \frac{1}{2e^t} - \frac{3}{2e^{3t}} \end{bmatrix} = \begin{bmatrix} \lim_{t\to\infty} \frac{1}{2e^t} + \lim_{t\to\infty} \frac{3}{2e^{3t}} \\ \lim_{t\to\infty} \frac{1}{2e^t} - \lim_{t\to\infty} \frac{3}{2e^{3t}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

<u>Remark:</u> In the example above we were lucky to find a basis of $\mathbb{R}^2$ formed of eigenvectors

of A, giving us some reassurance that we were probably able to find <u>all</u> the solutions of $\vec{y}'(t) = A\vec{y}(t)$ from them. The reason why these give all solutions, and how to find all solutions in general, is a bit more subtle and outside the scope of this course. Essentially, what is happening is that the solutions of a system of differential equations form a <u>finite dimensional vector space</u>, with a basis having n elements when A is an $n\times n$ matrix. To find all the solutions of a system, we first find a basis, and then we take linear combinations of the basis elements.

All in all, if A is an $n\times n$ diagonalizable matrix, the equation $\vec{y}'(t) = A\vec{y}(t)$ has solution:

$$\vec{y}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$$

where $(\lambda_1, \vec{v}_1), \ldots, (\lambda_n, \vec{v}_n)$ are linearly independent eigenvalue-eigenvector pairs of A and $c_1, \ldots, c_n$ are constant scalars. This works perfectly well for complex numbers.

<u>Example:</u> Solve the system of differential equations:

$$\begin{cases} y_1'(t) = -y_1(t) + 2y_2(t) \\ y_2'(t) = -2y_1(t) - y_2(t) \end{cases}$$ and find its stable state.

In matrix form, the system is:

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad \text{say} \quad A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}.$$

We first find the eigenvectors and eigenvalues of A:

$$\det(A - \lambda \cdot Id_2) = \det \begin{bmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{bmatrix} = \lambda^2 + 1 + 2\lambda + 4 = \lambda^2 + 2\lambda + 5 = (\lambda + 1 - 2i)(\lambda + 1 + 2i)$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \begin{cases} \dfrac{-2 + \sqrt{-16}}{2} = -1 + 2i. \\[2ex] \dfrac{-2 - \sqrt{-16}}{2} = -1 - 2i. \end{cases}$$

For $\lambda_1 = -1 + 2i$:

$$\vec{0} = (A - \lambda_1 Id_2)\vec{x} = \begin{bmatrix} -1+1-2i & 2 \\ -2 & -1+1-2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{so} \quad \vec{v_1} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

because $c_1 + i \cdot c_2 = \vec{0}$.

For $\lambda_2 = -1 - 2i$:

$$\vec{0} = (A - \lambda_2 Id_2)\vec{x} = \begin{bmatrix} -1+1+2i & 2 \\ -2 & -1+1+2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{so} \quad \vec{v_1} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

because $c_1 + (-i) \cdot c_2 = \vec{0}$.

The general solution of $\vec{y}'(t) = A\vec{y}(t)$ is then:

$$\vec{y}(t) = c_1 \cdot e^{(-1+2i)t} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 \cdot e^{(-1-2i)t} \cdot \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \text{for all scalars } c_1 \text{ and } c_2,$$

and the stable state is:

$$\lim_{t \to \infty} \vec{y}(t) = \lim_{t \to \infty} c_1 \cdot \underbrace{\frac{1}{e^t}}_{0} \cdot \underbrace{e^{2it}}_{|e^{2it}| = 1.} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} + \lim_{t \to \infty} c_2 \cdot \underbrace{\frac{1}{e^t}}_{0} \cdot \underbrace{\frac{1}{e^{2it}}}_{|e^{2it}| = 1.} \cdot \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

These have complex scalars, but we started with real scalars. To recover real scalars, we check if the real and imaginary components of the solution are linearly independent (in which case they form a basis of the solutions). Doing this in the general solution is very cumbersome (try simplifying $\mathrm{Re}(\vec{y}(t)) = \dfrac{\vec{y}(t) + \overline{\vec{y}(t)}}{2}$ and $\mathrm{Im}(\vec{y}(t)) = \dfrac{\vec{y}(t) - \overline{\vec{y}(t)}}{2i}$), so we look at the real and imaginary components of the basis $e^{(-1+2i)t} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $e^{(-1-2i)t} \cdot \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Since $e^{(-1+2i)t} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} e^{-t} \cdot e^{2it} \\ e^{-t} \cdot i \cdot e^{2it} \end{bmatrix} = \begin{bmatrix} e^{-t} \cdot (\cos(2t) + i\sin(2t)) \\ e^{-t} \cdot (i\cos(2t) - \sin(2t)) \end{bmatrix}$ then:

$$\mathrm{Re}\left(e^{(-1+2i)t} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix}\right) = \begin{bmatrix} e^{-t} \cdot \cos(2t) \\ -e^{-t} \cdot \sin(2t) \end{bmatrix} \text{ and } \mathrm{Im}\left(e^{(-1+2i)t} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix}\right) = \begin{bmatrix} e^{-t} \cdot \sin(2t) \\ e^{-t} \cdot \cos(2t) \end{bmatrix}.$$

Since $e^{(-1-2i)t} \cdot \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} e^{-t} \cdot e^{-2it} \\ e^{-t} \cdot (-i) \cdot e^{-2it} \end{bmatrix} = \begin{bmatrix} e^{-t} \cdot (\cos(-2t) + i\sin(-2t)) \\ e^{-t} \cdot (-i\cos(-2t) + \sin(-2t)) \end{bmatrix}$ then:

$$\mathrm{Re}\left(e^{(-1-2i)t} \cdot \begin{bmatrix} 1 \\ -i \end{bmatrix}\right) = \begin{bmatrix} e^{-t} \cdot \cos(-2t) \\ e^{-t} \cdot \sin(-2t) \end{bmatrix} = \begin{bmatrix} e^{-t} \cdot \cos(2t) \\ -e^{-t} \cdot \sin(2t) \end{bmatrix} \text{ and}$$

$$\mathrm{Im}\left(e^{(-1-2i)t} \cdot \begin{bmatrix} 1 \\ -i \end{bmatrix}\right) = \begin{bmatrix} e^{-t} \cdot \sin(-2t) \\ -e^{-t} \cdot \cos(-2t) \end{bmatrix} = -\begin{bmatrix} e^{-t} \cdot \sin(2t) \\ e^{-t} \cdot \cos(2t) \end{bmatrix}.$$

Then the basis vectors are a linear combination of $\begin{bmatrix} e^{-t} \cdot \cos(2t) \\ -e^{-t} \cdot \sin(2t) \end{bmatrix}$ and $\begin{bmatrix} e^{-t} \cdot \sin(2t) \\ e^{-t} \cdot \cos(2t) \end{bmatrix}$:

$$e^{(-1+2i)t} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} e^{-t} \cdot \cos(2t) \\ -e^{-t} \cdot \sin(2t) \end{bmatrix} + i \cdot \begin{bmatrix} e^{-t} \cdot \sin(2t) \\ e^{-t} \cdot \cos(2t) \end{bmatrix} \text{ and}$$

$$e^{(-1-2i)t} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} e^{-t} \cdot \cos(2t) \\ -e^{-t} \cdot \sin(2t) \end{bmatrix} - i \cdot \begin{bmatrix} e^{-t} \cdot \sin(2t) \\ e^{-t} \cdot \cos(2t) \end{bmatrix}.$$

This means that the general solution of $\vec{y}'(t) = A\vec{y}(t)$ can be written as:

$$\vec{y}(t) = c_1 \cdot \begin{bmatrix} e^{-t} \cdot \cos(2t) \\ -e^{-t} \cdot \sin(2t) \end{bmatrix} + c_2 \cdot \begin{bmatrix} e^{-t} \cdot \sin(2t) \\ e^{-t} \cdot \cos(2t) \end{bmatrix} \quad \text{for all scalars } c_1 \text{ and } c_2.$$

This phenomenon will always occur: if $(\lambda, \vec{v})$ is an eigenvalue-eigenvector pair of A, then their conjugate $(\bar{\lambda}, \bar{\vec{v}})$ is also an eigenvalue-eigenvector pair of A. If A is a 2x2 matrix, the general solution of the system of differential equations $\vec{y}'(t) = A\vec{y}(t)$ has two equivalent expressions:

$$\vec{y}(t) = c_1 \cdot e^{\lambda t} \cdot \vec{v} + c_2 \cdot e^{\bar{\lambda} t} \cdot \bar{\vec{v}} \quad \text{, its complex form, and}$$

$$\vec{y}(t) = c_1 \cdot \mathrm{Re}\left(e^{\lambda t} \cdot \vec{v}\right) + c_2 \cdot \mathrm{Im}\left(e^{\lambda t} \cdot \vec{v}\right) \quad \text{, its real form.}$$

If A is <u>not</u> diagonalizable, solving $\vec{y}'(t) = A\vec{y}(t)$ is much harder. An important thing to keep in mind is that the functions $e^{\lambda t}$ and $te^{\lambda t}$ are linearly independent, so if A only has one linearly independent eigenvalue-eigenvector pair $(\lambda, \vec{v})$ we not just have $\vec{y}(t) = e^{\lambda t}\vec{v}$ as a solution, but we may also have summands of the form $te^{\lambda t}\vec{w}$ for other vector $\vec{w}$.

<u>Example:</u> Solve the system of differential equations:

$$\begin{cases} y_1'(t) = y_1(t) + y_2(t) \\ y_2'(t) = y_2(t). \end{cases}$$

In matrix form, the system is:

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ \underline{not} diagonalizable.}$$

Thankfully, we can solve these equations one by one:

$y_2'(t) = y_2(t)$ has solution $y_2(t) = c_2 e^t$ for c_2 some constant scalar.

Substituting this into the first gives:

$$y_1'(t) = y_1(t) + y_2(t) = y_1(t) + c_2 e^t.$$

This is now a differential equation that can be solved using an <u>integrating factor.</u>

Without entering into details, the idea is to rewrite it as a derivative of a product. We can rewrite it as:

$$c_2 e^t = y_1'(t) - y_1(t) = y_1'(t) + \underbrace{(-1)}\cdot y_1(t)$$

to use the derivative of a product we multiply by $f(t) = e^{\int \overbrace{(-1)} dt} = e^{-t}$ to obtain:

$$c_2 = c_2 e^{-t+t} = e^{-t} \cdot c_2 \cdot e^t = e^{-t} y_1'(t) - e^{-t} y_1(t) = \frac{d}{dt}\left(e^{-t} y_1(t)\right)$$

which can now be integrated:

$$e^{-t} y_1(t) = \int \frac{d}{dt}\left(e^{-t} y_1(t)\right) dt = \int c_2 \, dt = c_2 t + \underbrace{c_1}_{\text{integration constant}}$$

for some constant scalar c_1. Simplifying:

$$y_1(t) = c_1 e^t + c_2 t e^t.$$

The solutions are $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 t e^t \\ c_2 e^t \end{bmatrix}$.

Although <u>not</u> all systems of differential equations will be decoupled like the previous one, if A is <u>not</u> diagonalizable then it decomposes as $A = BJB^{-1}$ where $J = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$. This is called the <u>Jordan normal form</u> and is beyond the scope of the course, but such a decomposition can be used to solve $\vec{y}'(t) = A\vec{y}(t)$.

<u>Example:</u> Solve the system of differential equations $\vec{y}'(t) = A\vec{y}(t)$ for $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ knowing that $A = BJB^{-1}$ for $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

The system is <u>not</u> decoupled:

$$\begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = -y_1(t) + 2y_2(t) \end{cases}$$ and writing $$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

does <u>not</u> seem to help because $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ is <u>not</u> diagonalizable. To confirm this, note:

$$\det(A - \lambda \cdot \mathrm{Id}_2) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

For $\lambda_1 = \lambda_2 = 1$ we have:

$$\vec{0} = (A - \lambda_i Id_2)\vec{x} = \begin{bmatrix} -1 & 1 \\ -1 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ with augmented matrix

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} \quad \text{and} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is the only linearly independent eigenvector of A. However, if we can solve:

$$\vec{z}\,'(t) = J\vec{z}(t) \quad \text{namely} \quad \begin{bmatrix} z_1'(t) \\ z_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \quad \text{namely} \quad \begin{cases} z_1'(t) = z_1(t) + z_2(t) \\ z_2'(t) = z_2(t) \end{cases}$$

then $\vec{y}(t) = B\vec{z}(t)$ will be the solution of $\vec{y}\,'(t) = A\vec{y}(t)$ because:

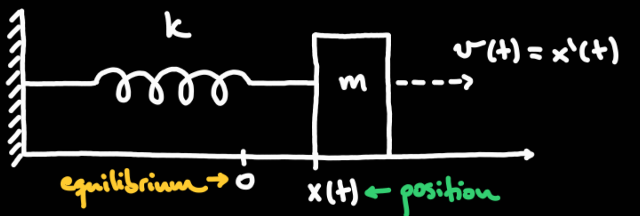
$$\vec{y}\,'(t) = \frac{d}{dt}\left(B\vec{z}(t)\right) = B\frac{d}{dt}\left(\vec{z}(t)\right) = B\vec{z}\,'(t) = BJ\vec{z}(t) \quad \text{coincides with}$$

$$A\vec{y}(t) = (BJB^{-1})\left(B\vec{z}(t)\right) = BJB^{-1}B\vec{z}(t) = BJ\vec{z}(t).$$

We found that the solution of $\vec{z}\,'(t) = J\vec{z}(t)$ is $\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 t e^t \\ c_2 e^t \end{bmatrix}$, so the solution of $\vec{y}\,'(t) = A\vec{y}(t)$ is $\vec{y}(t) = B\vec{z}(t)$ as below:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^t + c_2 t e^t \\ c_2 e^t \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 t e^t \\ (c_1 + c_2)e^t + c_2 t e^t \end{bmatrix}.$$

A practical application of these techniques is to find the equation of movement of a damped spring without friction:



Drag: $-bv(t)$

Spring: $-kx(t)$

so the equation of motion is $F(t) = m\,a(t)$:

$$m\,a(t) = -b\,v(t) - k\,x(t) \quad \text{namely} \quad m\,x''(t) + b\,x'(t) + k\,x(t) = 0.$$

Using $v(t) = x'(t)$ and $v'(t) = x''(t) = a(t)$ we can transform that second order differential equation into a system of linear differential equations:

$$\begin{cases} x'(t) = v(t) \\ v'(t) = -\frac{k}{m}\,x(t) - \frac{b}{m}\,v(t). \end{cases}$$

6.5.4. The matrix exponential and differential equations.

Given A an $n\times n$ matrix, we define its exponential as:

$$e^{A} = \sum_{m=0}^{\infty} \frac{A^m}{m!} = Id_n + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \dots$$

and e^{A} is an $n\times n$ matrix. The exponential function $y(t) = e^{At}$ satisfies the differential equation $y'(t) = A\,y(t)$, in the same way as the real valued and complex valued exponential functions. This means that the differential equation $\vec{y}'(t) = A\,\vec{y}(t)$ with initial condition $\vec{y}(0) = \vec{x}$ has solution $\vec{y}(t) = \vec{x}\,e^{At}$. When A is diagonalizable, we can compute A using its decomposition $A = BDB^{-1}$ where B is invertible and D is diagonal:

$$e^{A} = \sum_{m=0}^{\infty} \frac{A^m}{m!} = \sum_{m=0}^{\infty} \frac{(BDB^{-1})^m}{m!} = \sum_{m=0}^{\infty} \frac{BD^mB^{-1}}{m!} = B\left(\sum_{m=0}^{\infty} \frac{D^m}{m!}\right)B^{-1} = B\,e^{D}B^{-1};$$

and the exponential of a diagonal matrix is the exponential of its diagonal entries:

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \text{ gives } e^D = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}.$$

<u>Example:</u> Compute e^A for $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

We computed above the eigenvalue-eigenvector pairs of A as $\left(-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $\left(-3, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

so A is diagonalizable and $A = BDB^{-1}$ for $D = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Now:

$$e^A = B e^D B^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2e} + \frac{1}{2e^3} & \frac{1}{2e} - \frac{1}{2e^3} \\ \frac{1}{2e} - \frac{1}{2e^3} & \frac{1}{2e} + \frac{1}{2e^3} \end{bmatrix}.$$

When A is <u>not</u> diagonalizable the most systematic way of computing e^A relies on the Jordan normal form of A, which is outside the scope of the course. Essentially, the Jordan normal form of A can be used to write $A = S + N$ where S is a diagonalizable matrix, there exists a finite integer k such that $N^k = 0$, and $SN = NS$.

Then we can compute:

$$e^A = e^{S+N} \underset{\text{uses } SN=NS.}{=} \underbrace{e^S}_{\text{computable because } S \text{ diagonalizable.}} \underbrace{e^N}_{\text{computable because the sum } e^N = \sum_{m=0}^{\infty} \frac{N^m}{m!} \text{ is finite.}}$$

Example: Compute e^A for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Since we can decompose $A = S + N$ for $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $N^2 = 0$,

and $SN = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = NS$, then:

$$e^A = e^{S+N} = e^S e^N = \underbrace{\begin{bmatrix} e^1 & 0 \\ 0 & e^1 \end{bmatrix}}_{e^S} \cdot \underbrace{\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)}_{e^N} =$$

$$= \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & e \\ 0 & e \end{bmatrix}.$$

Example: Compute e^A for $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ knowing that $A = BJB^{-1}$ for $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

and $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

The diagonal part of J is $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the non-diagonal is $R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Now:

$$A = BJB^{-1} = B(D+R)B^{-1} = \underbrace{BDB^{-1}}_{S} + \underbrace{BRB^{-1}}_{N} = \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{S} + \overbrace{\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}}^{N} = S + N$$

with $N^2 = 0$ and $SN = N = NS$, so:

$$e^A = e^{S+N} = e^S e^N = \underbrace{\begin{bmatrix} e^1 & 0 \\ 0 & e^1 \end{bmatrix}}_{e^S} \cdot \underbrace{\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right)}_{e^N} =$$

$$= \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & e \\ -e & 2e \end{bmatrix}.$$