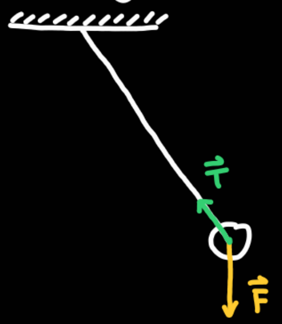
2. Vectors and geometry.

2.1. Motivation.

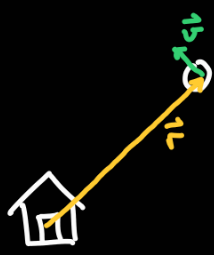
A <u>vector</u> is determined by a magnitude and a direction. They can be used to describe physical concepts such as: position, velocity, acceleration, force, and momentum.

A <u>scalar</u> is determined by a real (or complex) number. They can also be used to describe physical concepts, such as: length, mass, volume, time, and electric charge.

<u>Example:</u> The forces acting on a pendulum.



<u>Example:</u> The position of a cow in a field (with respect to the barn) and its velocity.

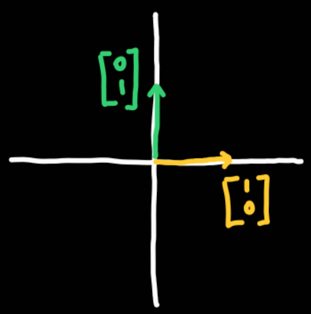


We will see that vectors can also describe the behavior of an electrical circuit and the behavior of a discrete dynamical system, among many others.

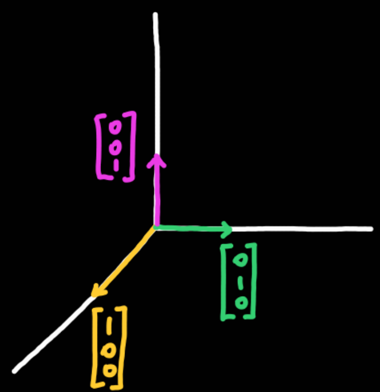
2.2. Vectors.

We express vectors as a list of scalars. Each position in the list corresponds to a predetermined direction, and each scalar corresponds to the contribution of that direction. We usually use the <u>standard basis vectors</u> as these predetermined directions. As a list of scalars, standard basis vectors have a one in exactly one position, and zeros in all other positions.

<u>Example:</u> The standard basis vectors in the plane are $\vec{e}_1 = \vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

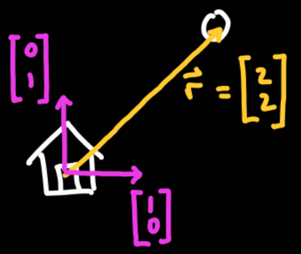


<u>Example:</u> The standard basis vectors in space are $\vec{e}_1 = \vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{e}_3 = \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

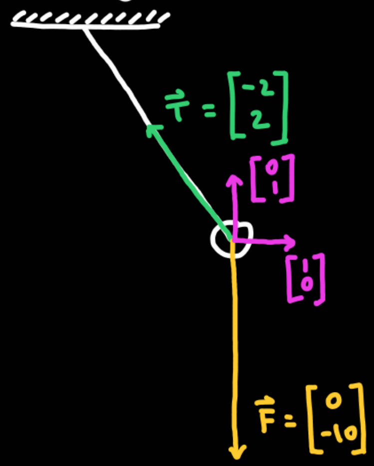


As we will later see, a vector with exactly n entries lies in an n-dimensional space.

Example: The position of a cow in a field (with respect to the barn).

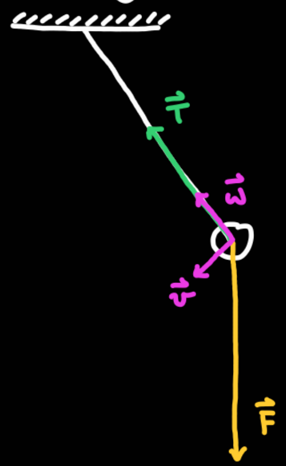


Example: The forces acting on a pendulum.



These preferred directions are called <u>coordinates</u>. Often it is useful to make different choices of coordinates, because it may help solving or setting up a problem.

Example: The forces acting on a pendulum.



To formalize the description of vectors in terms of coordinates, we use two operations.

① <u>Scalar multiplication:</u> Given a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and a scalar x, the scalar multiplication is the vector $x\vec{v} = \begin{bmatrix} x v_1 \\ x v_2 \end{bmatrix}$.

② <u>Vector addition:</u> Given vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, the vector addition is the vector $\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$.

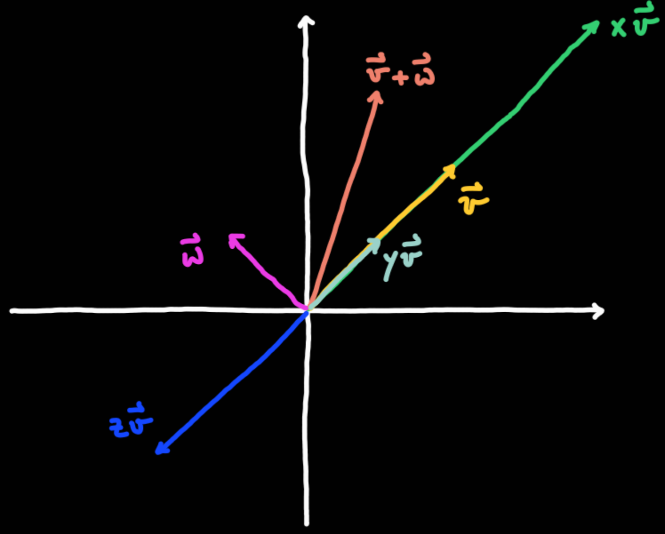
<u>Example:</u> Given $x = 2$, $y = \frac{1}{2}$, $z = -1$, $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then:

(i) $x\vec{v} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

(ii) $y\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(iii) $z\vec{v} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

(iv) $\vec{v} + \vec{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

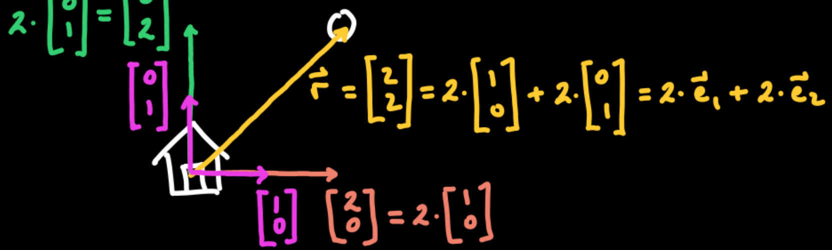


For n-dimensional vectors, these operations also occur in each entry.

Geometrically, $x\vec{v}$ is a vector in the direction of $\vec{v}$ with length x times the length of $\vec{v}$. Geometrically, $\vec{v} + \vec{w}$ is the long side of the triangle determined by doing first $\vec{v}$ and then $\vec{w}$. In this sense, we can intuitively understand vectors as giving the movement of a vehicle: scalar multiplication corresponds to a change in the speed, and vector addition corresponds to the combined effect of two consecutive displacements.

Expressing a vector in terms of coordinates is precisely saying how to get said vector from the coordinate vectors using scalar multiplication and vector addition.

<u>Example</u>: The position of a cow in a field (with respect to the barn).

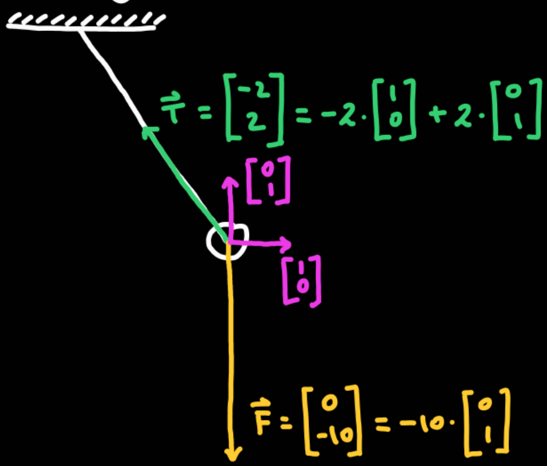


$$2\cdot\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix} \qquad \begin{bmatrix}0\\1\end{bmatrix}$$

$$\vec{r}=\begin{bmatrix}2\\2\end{bmatrix}=2\cdot\begin{bmatrix}1\\0\end{bmatrix}+2\cdot\begin{bmatrix}0\\1\end{bmatrix}=2\cdot\vec{e}_1+2\cdot\vec{e}_2$$

$$\begin{bmatrix}1\\0\end{bmatrix} \qquad \begin{bmatrix}2\\0\end{bmatrix}=2\cdot\begin{bmatrix}1\\0\end{bmatrix}$$

<u>Example</u>: The forces acting on a pendulum.

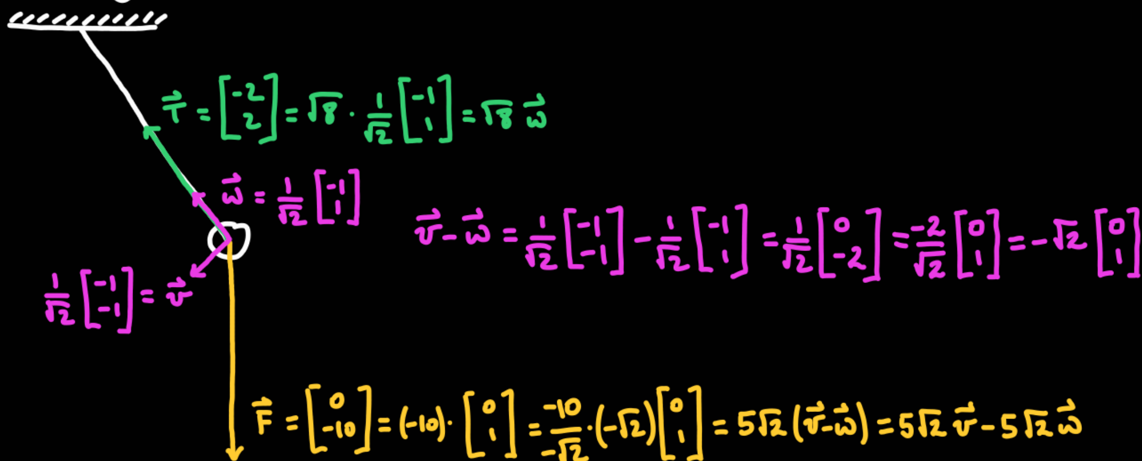


$$\vec{T}=\begin{bmatrix}-2\\2\end{bmatrix}=-2\cdot\begin{bmatrix}1\\0\end{bmatrix}+2\cdot\begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} \qquad \begin{bmatrix}1\\0\end{bmatrix}$$

$$\vec{F}=\begin{bmatrix}0\\-10\end{bmatrix}=-10\cdot\begin{bmatrix}0\\1\end{bmatrix}$$

<u>Example</u>: The forces acting on a pendulum.



$$\vec{T}=\begin{bmatrix}-2\\2\end{bmatrix}=\sqrt{8}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}=\sqrt{8}\,\vec{w}$$

$$\vec{w}=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\vec{v}-\vec{w}=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\-1\end{bmatrix}-\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\-2\end{bmatrix}=\frac{-2}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix}=-\sqrt{2}\begin{bmatrix}0\\1\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\-1\end{bmatrix}=\vec{v}$$

$$\vec{F}=\begin{bmatrix}0\\-10\end{bmatrix}=(-10)\cdot\begin{bmatrix}0\\1\end{bmatrix}=\frac{-10}{-\sqrt{2}}\cdot(-\sqrt{2})\begin{bmatrix}0\\1\end{bmatrix}=5\sqrt{2}\,(\vec{v}-\vec{w})=5\sqrt{2}\,\vec{v}-5\sqrt{2}\,\vec{w}$$

We will use scalar multiplication and vector addition for algebraic manipulations to solve equations, and it is important to know which properties they satisfy. Some are:

(i) $\vec{v}+\vec{w}=\vec{w}+\vec{v}$

(ii) $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$

(iii) $\vec{v}+\vec{0}=\vec{v}$, where $\vec{0}$ is the vector with all entries zero

(iv) $\vec{v}+(-\vec{v})=\vec{0}$

(v) $x(\vec{v}+\vec{w})=x\vec{v}+x\vec{w}$

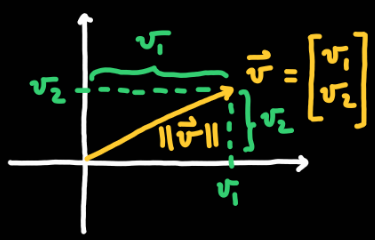
(vi) $(x+y)\vec{v}=x\vec{v}+y\vec{v}$

(vii) $(xy)\vec{v}=x(y\vec{v})$

(viii) $1\vec{v}=\vec{v}$

2.3. Geometrical aspects of vectors.

The magnitude of a vector $\vec{v}$ is given by its length, which we denote by $\|\vec{v}\|$.



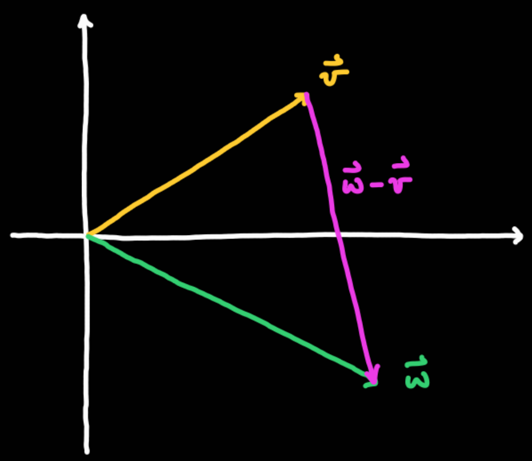
Given $\vec{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}$, by the Pythagoras theorem we have $\|\vec{v}\|^2=v_1^2+v_2^2$, so:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}.$$

In general, given $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ then:

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

The <u>distance</u> between two vectors is the distance between their targets.



The vector $\vec{w} - \vec{v}$ is precisely the missing side of the triangle given by $\vec{v}$ and $\vec{w}$, because $\vec{v} + (\vec{w} - \vec{v}) = \vec{w}$. Recall that vector addition corresponds to combining effects, so the equality $\vec{v} + (\vec{w} - \vec{v}) = \vec{w}$ is saying that "doing first $\vec{v}$ and then $\vec{w} - \vec{v}$ is the same as just doing $\vec{w}$". The distance between $\vec{v}$ and $\vec{w}$ is then:

$$\|\vec{v} - \vec{w}\|.$$

<u>Example</u>: Given $x = 2$, $y = \frac{-1}{2}$, $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then:

(i) The length of $y\vec{v} = \frac{-1}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is:

$$\|y\vec{v}\| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}.$$

(ii) The distance between $y\vec{v}$ and $x\vec{w}$ is:

$$y\vec{v} = \frac{-1}{2}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}, \quad x\vec{w} = 2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}, \quad x\vec{w} - y\vec{v} = \begin{bmatrix}-2\\2\end{bmatrix} - \begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$$

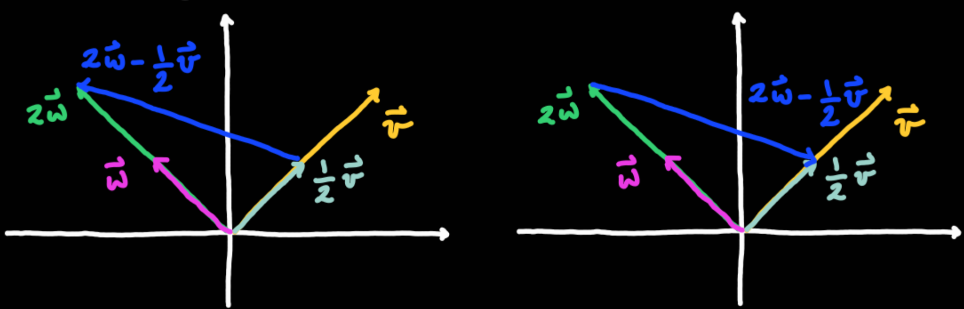
$$\|x\vec{w} - y\vec{v}\| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}.$$

The distance is also:

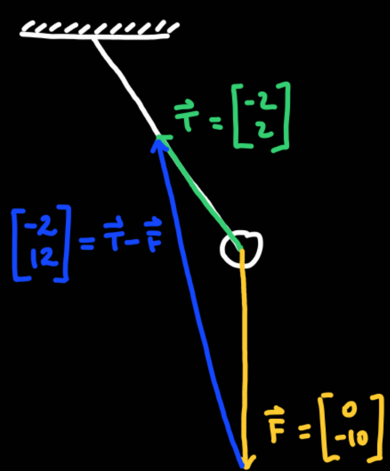
$$y\vec{v} - x\vec{w} = \begin{bmatrix}-1\\-1\end{bmatrix} - \begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\-3\end{bmatrix}, \quad \|y\vec{v} - x\vec{w}\| = \sqrt{1^2 + (-3)^2} = \sqrt{10}.$$

The equality $y\vec{v} - x\vec{w} = -(x\vec{w} - y\vec{v})$ says that the vectors $y\vec{v} - x\vec{w}$ and $x\vec{w} - y\vec{v}$ have the same magnitude but point in opposite directions.

Example: Graphically:



Example: The forces acting on a pendulum.



The distance between $\vec{F}$ and $\vec{T}$ is:

$$\vec{T} - \vec{F} = \begin{bmatrix}-2\\2\end{bmatrix} - \begin{bmatrix}0\\-10\end{bmatrix} = \begin{bmatrix}-2\\12\end{bmatrix}$$

$$\|\vec{T} - \vec{F}\| = \sqrt{(-2)^2 + 12^2} = \sqrt{148} = \sqrt{4 \cdot 37} = 2 \cdot \sqrt{37}$$

The dot product between $\vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}w_1\\w_2\end{bmatrix}$ is the scalar:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2.$$

In n-dimensions, the dot product of $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ is the scalar:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n .$$

It is important to be familiar with the properties they satisfy, such as:

(i) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

(ii) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

(iii) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

(iv) $x(\vec{v} \cdot \vec{w}) = (x\vec{v}) \cdot \vec{w}$

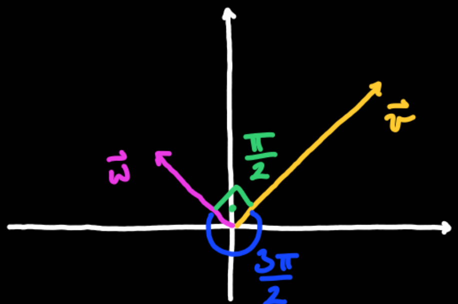
(v) $\vec{0} \cdot \vec{v} = \vec{0}$

(vi) $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos(\theta)$, where θ is the smallest angle between $\vec{v}$ and $\vec{w}$.

(vii) $\vec{v} \cdot \vec{w} = 0$ if and only if $\vec{v} = \vec{0}$ or $\vec{w} = 0$ or $\vec{v}$ and $\vec{w}$ are orthogonal.

We say that $\vec{v}$ and $\vec{w}$ are <u>orthogonal</u> when the smallest angle between them is $\frac{\pi}{2}$, that is, when $\vec{v}$ and $\vec{w}$ are perpendicular.

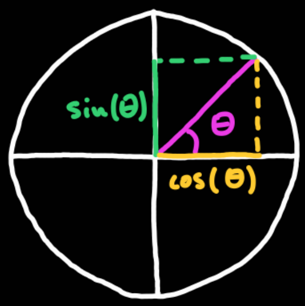
<u>Example:</u> Find the angle between $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.



We compute:

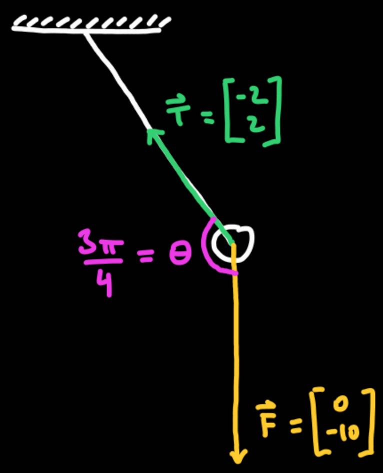
$$\vec{v}\cdot\vec{w} = \begin{bmatrix}2\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\1\end{bmatrix} = 2\cdot(-1)+2\cdot 1 = 0, \quad \|\vec{v}\| = \sqrt{2^2+2^2} = \sqrt{8}, \quad \|\vec{w}\| = \sqrt{(-1)^2+1^2} = \sqrt{2}$$

$$\cos(\theta) = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|} = \frac{0}{\sqrt{8}\cdot\sqrt{2}} = 0, \qquad \theta = \arccos(0) = \frac{\pi}{2}.$$



Example: The forces acting on a pendulum.

Find the angle between $\vec{T}$ and $\vec{F}$.



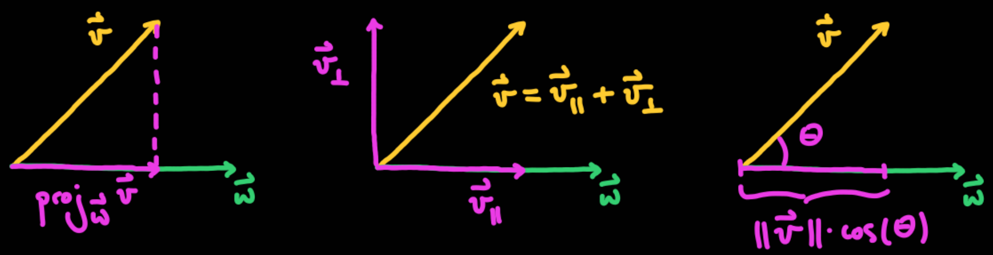
We compute:

$$\vec{T}\cdot\vec{F} = \begin{bmatrix}-2\\2\end{bmatrix}\cdot\begin{bmatrix}0\\-10\end{bmatrix} = (-2)\cdot 0 + 2\cdot(-10) = -20,$$

$$\|\vec{T}\| = \sqrt{(-2)^2+2^2} = \sqrt{8}, \quad \|\vec{F}\| = \sqrt{0^2+(-10)^2} = 10$$

$$\cos(\theta) = \frac{\vec{T}\cdot\vec{F}}{\|\vec{T}\|\cdot\|\vec{F}\|} = \frac{-20}{\sqrt{8}\cdot 10} = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}, \qquad \theta = \frac{3\pi}{4}.$$

The <u>projection</u> of $\vec{v}$ onto $\vec{w}$ is the component of $\vec{v}$ pointing in the direction of $\vec{w}$.



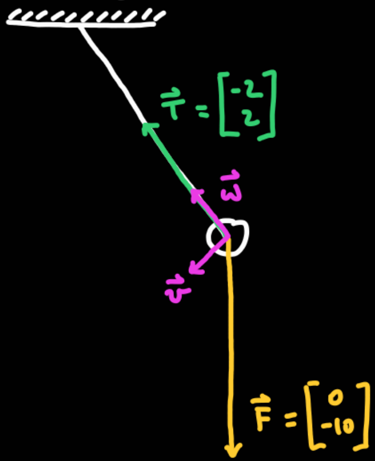
If the angle between $\vec{v}$ and $\vec{w}$ is θ, the length of $\mathrm{proj}_{\vec{w}}\vec{v}$ is $\|\vec{v}\|\cdot\cos(\theta)$.

The <u>unit vector</u> pointing in the direction of $\vec{w}$ is $\frac{\vec{w}}{\|\vec{w}\|}$, the vector of length one that points in the same direction as $\vec{w}$.

Putting the above together, the projection of $\vec{v}$ onto $\vec{w}$ is the vector of length $\|\vec{v}\| \cdot \cos(\theta)$ pointing in the direction of $\frac{\vec{w}}{\|\vec{w}\|}$. Since $\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}$ then:

$$\text{proj}_{\vec{w}} \vec{v} = \left(\|\vec{v}\| \cdot \cos(\theta)\right) \frac{\vec{w}}{\|\vec{w}\|} = \left(\frac{\|\vec{v}\| \, (\vec{v} \cdot \vec{w})}{\|\vec{v}\| \cdot \|\vec{w}\|}\right) \frac{\vec{w}}{\|\vec{w}\|} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \, \vec{w}.$$

<u>Example</u>: The forces acting on a pendulum.



(i) Find the unit vector $\vec{w}$ pointing in the direction of $\vec{T}$.

(ii) Find a unit vector $\vec{v}$ pointing in the direction orthogonal to $\vec{T}$.

(iii) Find the projection of $\vec{F}$ in the direction of $\vec{v}$.

(iv) Find the projection of $\vec{F}$ in the direction of $\vec{w}$.

(v) Find the projection of $\vec{T}$ in the direction of $\vec{w}$.

(vi) Find the projection of $\vec{T}$ in the direction of $\vec{v}$.

(i) The unit vector pointing in the direction of $\vec{T}$ is:

$$\vec{w} = \frac{\vec{T}}{\|\vec{T}\|} = \frac{\begin{bmatrix}-2\\2\end{bmatrix}}{\sqrt{8}} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}.$$

(ii) A unit vector $\vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$ orthogonal to $\vec{T}$ satisfies $\|\vec{v}\| = 1$ and $\vec{v}\cdot\vec{w} = 0$. So:

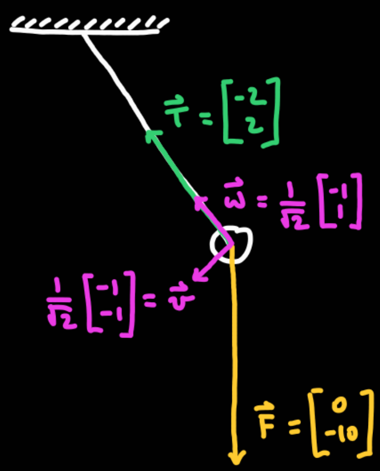
$$1 = \|\vec{v}\| = \sqrt{v_1^2 + v_2^2}, \quad \text{equivalently} \quad 1 = v_1^2 + v_2^2, \quad \text{and}$$

$$0 = \vec{v}\cdot\vec{w} = v_1\cdot\left(\frac{-1}{\sqrt{2}}\right) + v_2\cdot\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(-v_1 + v_2), \quad \text{equivalently} \quad v_1 = v_2.$$

Then the first equation becomes:

$$1 = v_1^2 + v_1^2 = 2v_1^2 \quad \text{so} \quad v_1^2 = \frac{1}{\sqrt{2}} \quad \text{and} \quad v_1 = \pm\frac{1}{\sqrt{2}},$$

giving two solutions. For $v_1 = \frac{1}{\sqrt{2}}$ we obtain $\vec{v} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, for $v_1 = \frac{-1}{\sqrt{2}}$ we obtain $\vec{v} = \begin{bmatrix}\frac{-1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\-1\end{bmatrix}$. Let's choose the latter (bot are valid).



(iii) We have:

$$\text{proj}_{\vec{v}}\vec{F} = \frac{\vec{v}\cdot\vec{F}}{\|\vec{v}\|^2}\vec{v} = \frac{\left(\frac{-1}{\sqrt{2}}\right)\cdot 0 + \left(\frac{-1}{\sqrt{2}}\right)\cdot(-10)}{1}\vec{v} = \frac{10}{\sqrt{2}}\vec{v} = 5\sqrt{2}\,\vec{v} = 5\sqrt{2}\,\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}-5\\-5\end{bmatrix}.$$

(iv) We have:

$$\operatorname{proj}_{\vec{w}} \vec{F} = \frac{\vec{w}\cdot\vec{F}}{\|\vec{w}\|^2}\vec{w} = \frac{\left(\frac{-1}{\sqrt{2}}\right)\cdot 0 + \frac{1}{\sqrt{2}}\cdot(-10)}{1}\vec{w} = \frac{-10}{\sqrt{2}}\vec{w} = -5\sqrt{2}\,\vec{w} = -5\sqrt{2}\,\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}5\\-5\end{bmatrix}.$$

(v) We have:

$$\operatorname{proj}_{\vec{w}} \vec{T} = \frac{\vec{w}\cdot\vec{T}}{\|\vec{w}\|^2}\vec{w} = \frac{\left(\frac{-1}{\sqrt{2}}\right)\cdot(-2) + \frac{1}{\sqrt{2}}\cdot 2}{1}\vec{w} = \left(\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right)\vec{w} = \sqrt{8}\,\vec{w} = \sqrt{8}\,\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix} = \vec{T}.$$

(vi) We have:

$$\operatorname{proj}_{\vec{v}} \vec{T} = \frac{\vec{v}\cdot\vec{T}}{\|\vec{v}\|^2}\vec{v} = \frac{\left(\frac{-1}{\sqrt{2}}\right)\cdot(-2) + \left(\frac{-1}{\sqrt{2}}\right)\cdot 2}{1}\vec{v} = \frac{0}{1}\vec{v} = 0\vec{v} = \vec{0}.$$