2.4. Determinants and the crossed product.

A <u>matrix</u> is a rectangular array of numbers. We say that M is an $n \times m$ matrix when it has n rows and m columns.

<u>Example:</u> (i) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a 2×2 square matrix.

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a 2×3 rectangular matrix.

(iii) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a 3×2 rectangular matrix.

(iv) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a 3×3 square matrix.

To M a square matrix we associate $\det(M)$ the <u>determinant</u> of M as follows.

(i) $M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

(ii) $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} =$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

(iii) $M = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11}\det\begin{bmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{bmatrix} + \cdots + (-1)^{n+1} a_{1n}\det\begin{bmatrix} a_{21} & \cdots & a_{2n-1} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn-1} \end{bmatrix}$.

Example: (i) $\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1\cdot 4 - 2\cdot 3 = 4-6 = -2.$

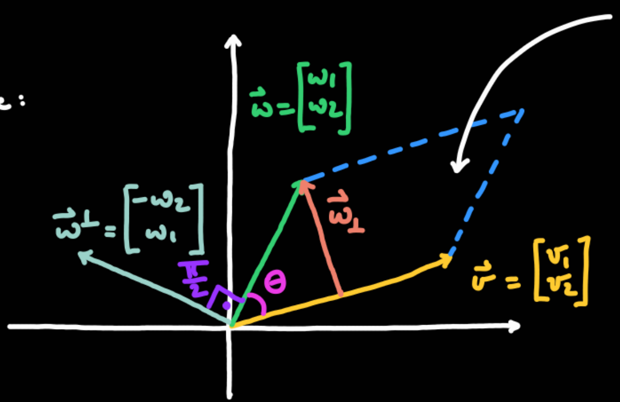
(ii) $\det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1\cdot\det\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2\det\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3\cdot\det\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} =$

$$= 5\cdot 9 - 6\cdot 8 - 2\cdot(4\cdot 9) - 2\cdot(-6\cdot 7) + 3\cdot(4\cdot 8) + 3\cdot(-5\cdot 7) =$$

$$= 45 - 48 - 72 + 84 + 96 - 105 = 0.$$

Geometrically, the absolute value of the determinant of a square matrix gives the volume of the parallelotope determined by its rows (equivalently, it also describes the volume of the parallelotope determined by its columns).

In two dimensions, we have:



$\text{Area} = \text{base} \cdot \text{height} =$

$= \|\vec{v}\| \cdot \|\vec{w}_{\perp}\| =$

$= \|\vec{v}\| \cdot \|\vec{w}\| \cdot |\sin(\theta)|$

$\|\vec{w}_{\perp}\| = \|\vec{w}\| \cdot |\sin(\theta)|$

where $\vec{w}_{\perp}$ is the component of $\vec{w}$ perpendicular to $\vec{v}$. Namely, since the component of $\vec{w}$ parallel to $\vec{v}$ is $\text{proj}_{\vec{v}}\,\vec{w}$, then $\vec{w}_{\perp} = \vec{w} - \text{proj}_{\vec{v}}\,\vec{w}$. The vectors $\vec{w}$ and $\vec{w}^{\perp}$ are perpendicular because $\vec{w}\cdot\vec{w}^{\perp} = w_1\cdot(-w_2) + w_2\cdot w_1 = 0$. The determinant of the matrix with rows $\vec{v}$ and $\vec{w}$ is:

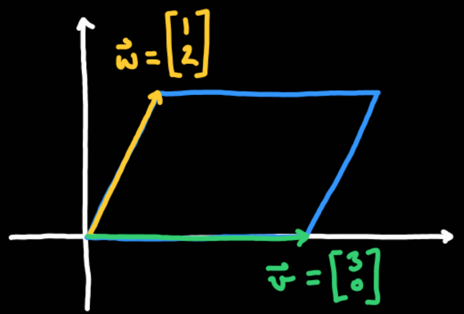
$$\left|\det\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}\right| = |v_1 w_2 - v_2 w_1| = |-v_1 w_2 + v_2 w_1| =$$

$$= |v_1\cdot(-w_2) + v_2\cdot w_1| = \left|\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\cdot\begin{bmatrix} -w_2 \\ w_1 \end{bmatrix}\right| = |\vec{v}\cdot\vec{w}^{\perp}| = \big|\,\|\vec{v}\|\cdot\|\vec{w}^{\perp}\|\cdot\cos(\alpha)\,\big| = \|\vec{v}\|\cdot\|\vec{w}^{\perp}\|\cdot|\cos(\alpha)| =$$

$$\underset{\|\vec{w}^{\perp}\| = \|\vec{w}\|}{=} \|\vec{v}\|\cdot\|\vec{w}\|\cdot|\cos(\alpha)| \underset{\alpha = \frac{\pi}{2}+\theta}{=} \|\vec{v}\|\cdot\|\vec{w}\|\cdot\left|\cos\left(\frac{\pi}{2}+\theta\right)\right| \underset{\cos\left(\frac{\pi}{2}+\theta\right) = -\sin(\theta)}{=} \|\vec{v}\|\cdot\|\vec{w}\|\cdot|-\sin(\theta)| = \|\vec{v}\|\cdot\|\vec{w}\|\cdot|\sin(\theta)|$$

which indeed coincides with the area computed above.

__Example:__ Find the area of the following parallelogram.

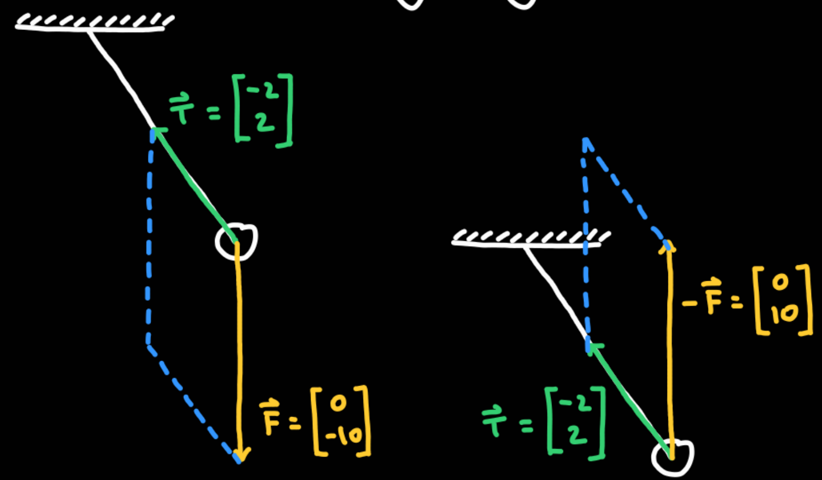


Geometrically, the area is: $3\cdot 2 = 6$ units.

The parallelogram is determined by $\vec{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so its area is:

$$\det\begin{bmatrix} - \vec{v} - \\ - \vec{w} - \end{bmatrix} = \det\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} = 3\cdot 2 - 0\cdot 1 = 6.$$

__Example:__ Find the area of the parallelogram given by $\vec{T}$ and $\vec{F}$, and by $\vec{T}$ and $-\vec{F}$.



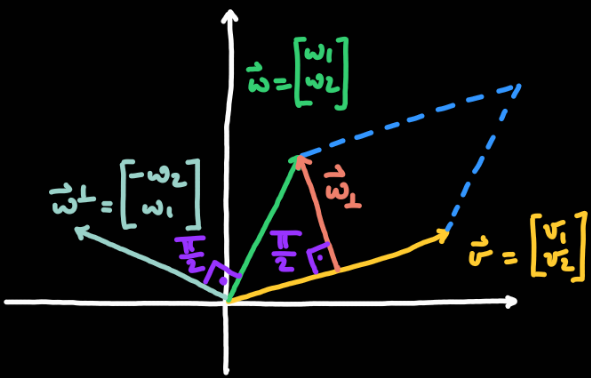
The area is the absolute value of the determinant of the matrix whose first row is $\vec{T}$ and whose second row is $\vec{F}$ or $-\vec{F}$:

$$\det\begin{bmatrix} - & \vec{T} & - \\ - & \vec{F} & - \end{bmatrix} = \det\begin{bmatrix} -2 & 2 \\ 0 & -10 \end{bmatrix} = (-2)\cdot(-10) - 2\cdot 0 = 20.$$

$$\det\begin{bmatrix} - & \vec{T} & - \\ - & -\vec{F} & - \end{bmatrix} = \det\begin{bmatrix} -2 & 2 \\ 0 & 10 \end{bmatrix} = (-2)\cdot 10 - 2\cdot 0 = -20.$$

The area is 20 units.

A key tool in this geometric interpretation was our ability to build vectors that are perpendicular to other vectors.



Vectors $\vec{w}$ and $\vec{w}^{\perp}$ are orthogonal.

Vectors $\vec{v}$ and $\vec{w}_{\perp}$ are orthogonal.

To interpret the determinant of a 3x3 matrix, it will also be useful to know how to obtain perpendicular vectors in three dimensions. The <u>crossed product</u> of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ is the vector $\vec{v} \times \vec{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$. A mnemonic to compute it is:

$$\vec{v} \times \vec{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} = \vec{e}_1 (v_2 w_3 - v_3 w_2) + \vec{e}_2 (v_3 w_1 - v_1 w_3) + \vec{e}_3 (v_1 w_2 - v_2 w_1) =$$

$$= \vec{e}_1 (v_2 w_3 - v_3 w_2) - \vec{e}_2 (v_1 w_3 - v_3 w_1) + \vec{e}_3 (v_1 w_2 - v_2 w_1) =$$

$$= \vec{e}_1 \det\begin{bmatrix} v_2 & v_3 \\ w_2 & w_3 \end{bmatrix} - \vec{e}_2 \det\begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix} + \vec{e}_3 \det\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} =$$

$$= \det\begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

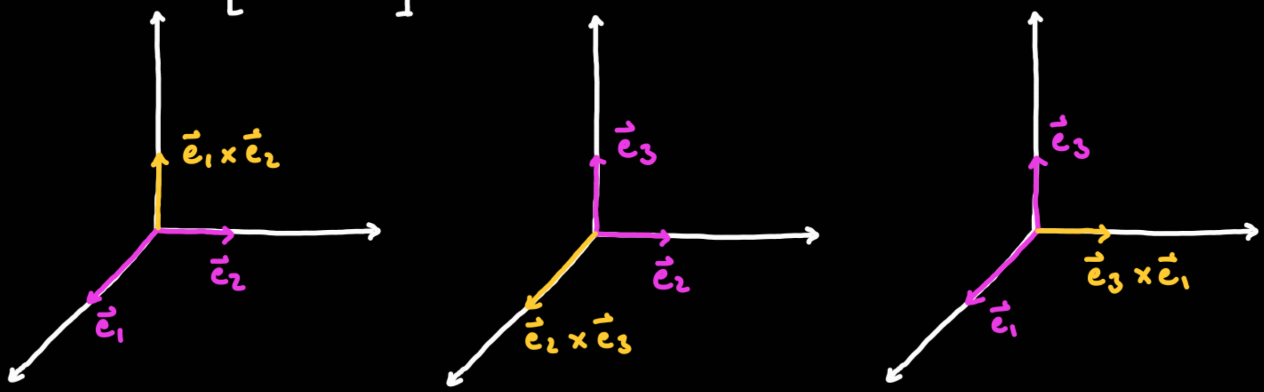
where it is important to keep in mind that the last equality and determinant are <u>not</u> well defined. It does <u>not</u> make sense to take the determinant of a matrix whose entries are scalars and vectors.

<u>Example:</u> Compute the following crossed products:

(i) $\vec{e}_1 \times \vec{e}_2 = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \vec{e}_1 \cdot (0\cdot 0 - 0 \cdot 1) - \vec{e}_2 \cdot (1 \cdot 0 - 0 \cdot 0) + \vec{e}_3 \cdot (1 \cdot 1 - 0 \cdot 0) = \vec{e}_3$

(ii) $\vec{e}_2 \times \vec{e}_3 = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \vec{e}_1 \cdot (1\cdot 1 - 0 \cdot 0) - \vec{e}_2 \cdot (0 \cdot 1 - 0 \cdot 0) + \vec{e}_3 \cdot (0 \cdot 0 - 1 \cdot 0) = \vec{e}_1$

(iii) $\vec{e}_3 \times \vec{e}_1 = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \vec{e}_1 \cdot (0\cdot 0 - 1 \cdot 0) - \vec{e}_2 \cdot (0 \cdot 0 - 1 \cdot 1) + \vec{e}_3 \cdot (0 \cdot 0 - 0 \cdot 1) = \vec{e}_2$



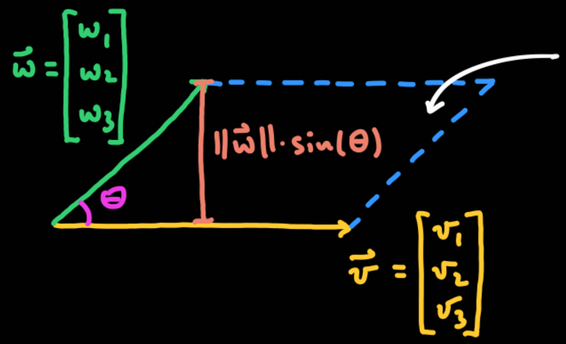
Geometric properties of the crossed product:

(i) $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ and to $\vec{w}$

(ii) $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin(\theta)$, where θ is the smallest angle between $\vec{v}$ and $\vec{w}$.

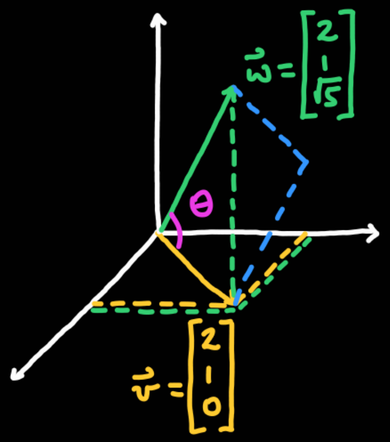
(iii) $\vec{v}$, $\vec{w}$, and $\vec{v} \times \vec{w}$ follow the right hand rule.

In particular, $\|\vec{v} \times \vec{w}\|$ is the area of the parallelogram given by $\vec{v}$ and $\vec{w}$.



Area = base · height $= \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin(\theta)$

<u>Example:</u> Find the area of the parallelogram given by $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ \sqrt{5} \end{bmatrix}$.



We have:

$$\|\vec{v}\| = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{5}$$

$$\|\vec{w}\| = \sqrt{2^2 + 1^2 + 5} = \sqrt{10}$$

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}} \quad \text{so} \quad \theta = \frac{\pi}{4}$$

The area is:

$$\|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin(\theta) = \sqrt{5} \cdot \sqrt{10} \cdot \sin\left(\frac{\pi}{4}\right) = \sqrt{50} \cdot \frac{1}{\sqrt{2}} = \sqrt{25} = 5 \text{ units.}$$

We also have:

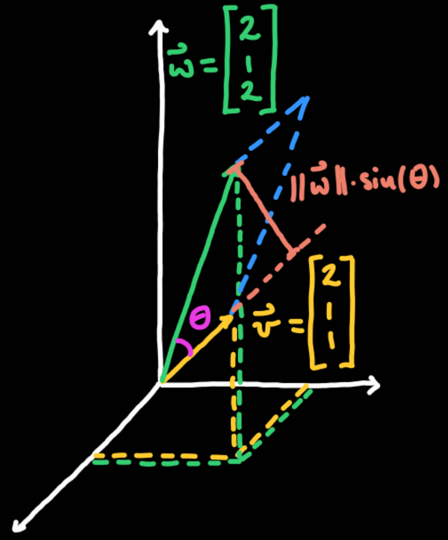
$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 2 & 1 & 0 \\ 2 & 1 & \sqrt{5} \end{bmatrix} = \sqrt{5}\,\vec{e}_1 - 2\sqrt{5}\,\vec{e}_2 + 0 \cdot \vec{e}_3 = \sqrt{5} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\sqrt{5} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ -2\sqrt{5} \\ 0 \end{bmatrix}$$

and the area is its length:

$$\|\vec{v} \times \vec{w}\| = \sqrt{5 + (-2\sqrt{5})^2 + 0^2} = \sqrt{5 + 4 \cdot 5} = \sqrt{25} = 5 \text{ units.}$$

A key point is that we do <u>not</u> need to know the angle between $\vec{v}$ and $\vec{w}$ to compute $\vec{v} \times \vec{w}$, so we can find areas without using trigonometry.

Example: Find the area of the parallelogram given by $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.



We first compute:

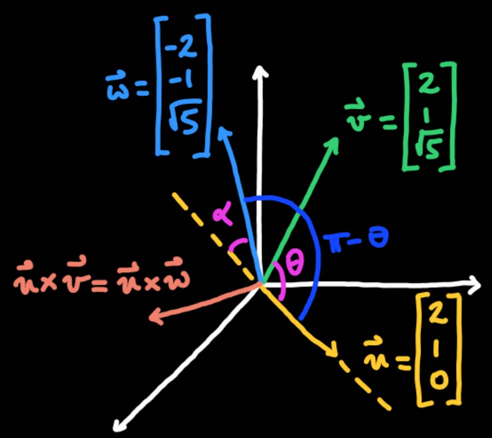
$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \vec{e}_1 - 2\vec{e}_2 + 0\vec{e}_3 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

So the area is:

$$\|\vec{v} \times \vec{w}\| = \sqrt{1^2 + (-2)^2 + 0^2} = \sqrt{5} \text{ units.}$$

Although both $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos(\theta)$ and $\vec{v} \times \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin(\theta)$ express a relation between $\vec{v}$, $\vec{w}$, and θ the smallest angle between them, we can <u>not</u> use the cross product to find θ. This is because $\sin(\theta) = \sin(\pi - \theta)$.

Example: For $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ \sqrt{5} \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -2 \\ -1 \\ \sqrt{5} \end{bmatrix}$, compute the following.



(i) $\vec{u} \cdot \vec{v} = 2 \cdot 2 + 1 \cdot 1 + 0 \cdot \sqrt{5} = 5$

(ii) $\vec{u} \cdot \vec{w} = 2 \cdot (-2) + 1 \cdot (-1) + 0 \cdot \sqrt{5} = -5$

(iii) $\vec{u} \times \vec{v} = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 2 & 1 & 0 \\ 2 & 1 & \sqrt{5} \end{bmatrix} = \sqrt{5}\,\vec{e}_1 - 2\sqrt{5}\,\vec{e}_2 + 0\vec{e}_3 = \begin{bmatrix} \sqrt{5} \\ -2\sqrt{5} \\ 0 \end{bmatrix}$

(iv) $\vec{u} \times \vec{w} = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 2 & 1 & 0 \\ -2 & -1 & \sqrt{5} \end{bmatrix} = \sqrt{5}\,\vec{e}_1 - 2\sqrt{5}\,\vec{e}_2 + 0\vec{e}_3 = \begin{bmatrix} \sqrt{5} \\ -2\sqrt{5} \\ 0 \end{bmatrix}$

We can tell that the angle between $\vec{u}$ and $\vec{v}$ is:

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}\right) = \arccos\left(\frac{5}{\sqrt{50}}\right) = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

and that the angle between $\vec{u}$ and $\vec{w}$ is:

$$\alpha = \arccos\left(\frac{\vec{u}\cdot\vec{w}}{\|\vec{u}\|\cdot\|\vec{w}\|}\right) = \arccos\left(\frac{-5}{\sqrt{50}}\right) = \arccos\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}.$$

Then $\pi - \theta = \pi - \frac{\pi}{4} = \frac{4\pi}{4} - \frac{\pi}{4} = \frac{3\pi}{4} = \alpha$. Indeed:

$$\sin(\theta) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\left(\frac{3\pi}{4}\right) = \sin(\alpha)$$

and the cross product cannot distinguish between θ and α.

Algebraic properties of the cross product:

(i) $\vec{v}\times\vec{w} = -\vec{w}\times\vec{v}$

(ii) $\vec{u}\times(\vec{v}\times\vec{w}) = -(\vec{u}\cdot\vec{v})\vec{w} + (\vec{u}\cdot\vec{w})\vec{v}$ and $(\vec{v}\times\vec{w})\times\vec{u} = (\vec{v}\cdot\vec{u})\vec{w} - (\vec{w}\cdot\vec{u})\vec{v}$

(iii) $y(\vec{v}\times\vec{w}) = (y\vec{v})\times\vec{w}$

(iv) $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$

(v) $\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w}$

We can now verify the geometric interpretation of the absolute value of a determinant in three dimensions as the volume of the parallepiped given by the rows. Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$, θ the angle between $\vec{u}$ and $\vec{v}\times\vec{w}$, and α the angle between $\vec{v}$ and $\vec{w}$, so:

$$\left|\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right| = \left| u_1 \det\begin{bmatrix} v_2 & v_3 \\ w_2 & w_3 \end{bmatrix} - u_2 \det\begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix} + u_3 \det\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} \right| =$$

$$= \left| u_1 (v_2 w_3 - v_3 w_2) - u_2 (v_1 w_3 - v_3 w_1) + u_3 (v_1 w_2 - v_2 w_1) \right| =$$

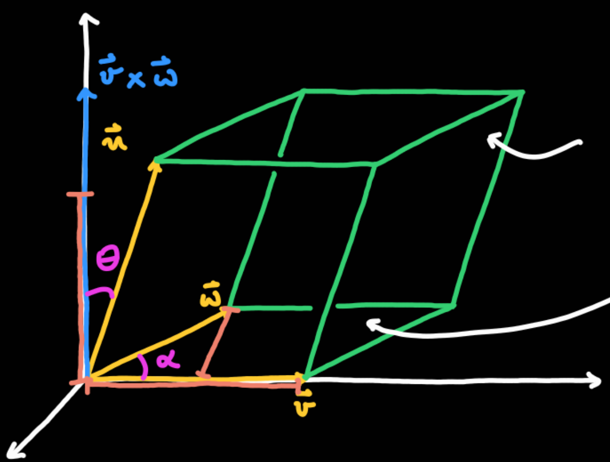
$$= \left| u_1 (v_2 w_3 - v_3 w_2) + u_2 (v_3 w_1 - v_1 w_3) + u_3 (v_1 w_2 - v_2 w_1) \right| =$$

$$= \left| \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} \right| = \left| \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right) \right| = |\vec{u} \cdot (\vec{v} \times \vec{w})| =$$

$$= \|\vec{u}\| \cdot \|\vec{v} \times \vec{w}\| \cdot |\cos(\theta)| = \|\vec{u}\| \cdot |\cos(\theta)| \cdot \|\vec{v} \times \vec{w}\| =$$

$$= \|\vec{u}\| \cdot |\cos(\theta)| \cdot \|\vec{u}\| \cdot \|\vec{w}\| \cdot |\sin(\alpha)|$$

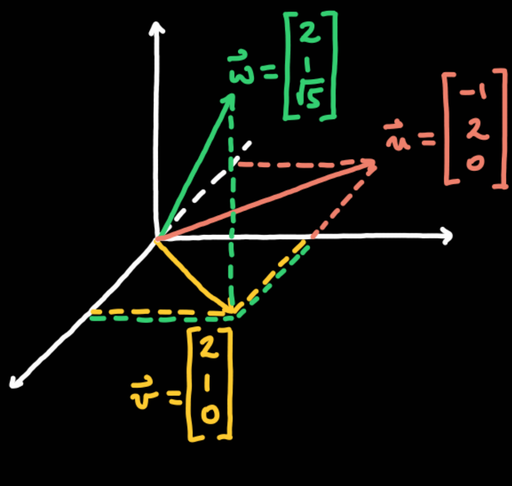
which is exactly the claimed volume.



Volume = base · height $= \|\vec{v}\| \cdot \|\vec{w}\| \cdot |\cos(\alpha)| \cdot \|\vec{u}\| \cdot |\sin(\theta)|$

Area = base · height $= \|\vec{v}\| \cdot \|\vec{w}\| \cdot |\cos(\alpha)|$

<u>Example:</u> Find the volume of the parallepiped given by $\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ \sqrt{5} \end{bmatrix}$.



Using the geometric interpretation of the determinant, the area is:

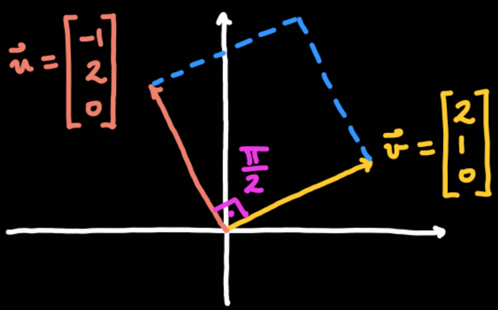
$$\left|\det\begin{bmatrix} -1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & \sqrt{5} \end{bmatrix}\right| = \left| (-1) \cdot 1 \cdot \sqrt{5} - 2 \cdot 2 \cdot \sqrt{5} \right| = 5\sqrt{5} \text{ units.}$$

We can check this by hand, multiplying the area of the base by the height.

The base is formed by the vectors $\vec{u}$ and $\vec{v}$. Since:

$$\vec{u}\cdot\vec{v} = (-1)\cdot 2 + 2\cdot 1 + 0\cdot 0 = 0$$

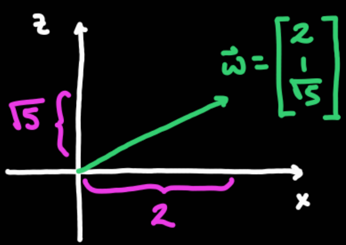
then $\vec{u}$ and $\vec{v}$ are orthogonal. Moreover, the vectors $\vec{v}$ and $\vec{u}$ have third component zero, so they are in the x-y-plane. We can draw them as follows.

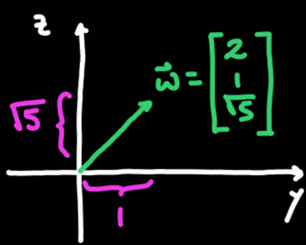


The area of the base is then the area of the rectangle having $\vec{u}$ and $\vec{v}$ for sides, namely:

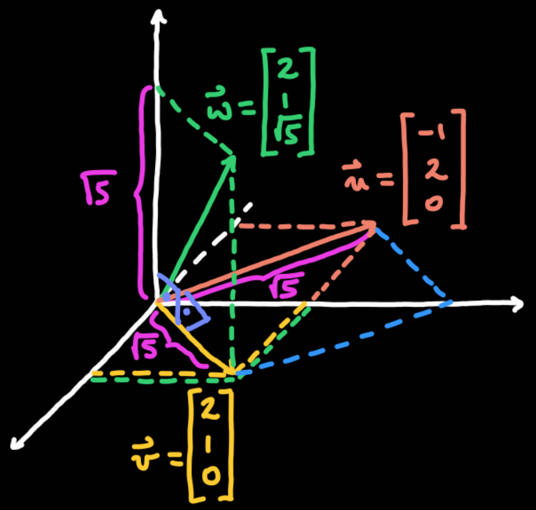
$$\|\vec{u}\|\cdot\|\vec{v}\| = \sqrt{(-1)^2+2^2+0^2}\cdot\sqrt{2^2+1^2+0^2} = \sqrt{5}\cdot\sqrt{5} = 5.$$

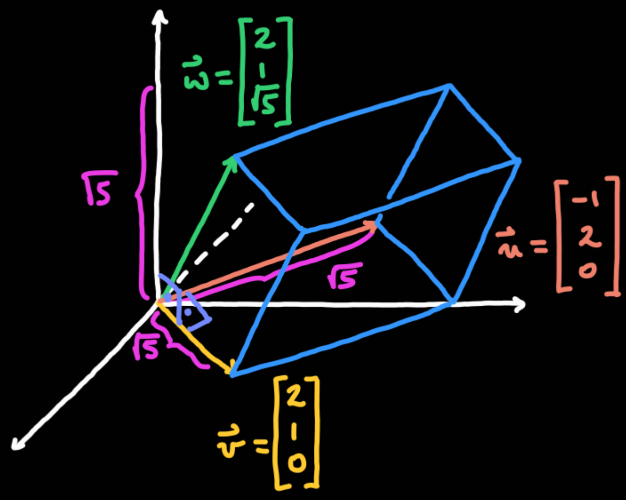
Since $\vec{u}$ and $\vec{v}$ are in the x-y-plane, the direction orthogonal to both of them is the z-axis. The component of $\vec{w}$ orthogonal to both $\vec{u}$ and $\vec{v}$ is then the component of $\vec{w}$ in the z-axis, namely the third component of $\vec{w}$.





We can draw the following.



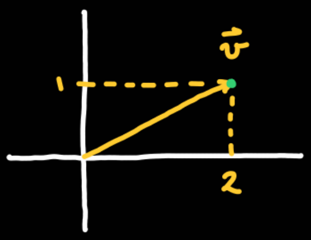


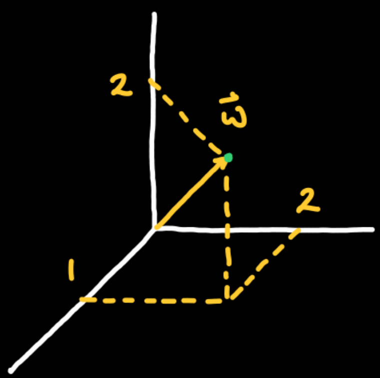
So the total volume is: $\text{Volume} = \text{base} \cdot \text{height} = 5\sqrt{5}$ units.

2.5. Lines and planes.

A <u>point</u> is a location in a plane or in space. A point can be understood as the the target of a vector whose source is at the origin.

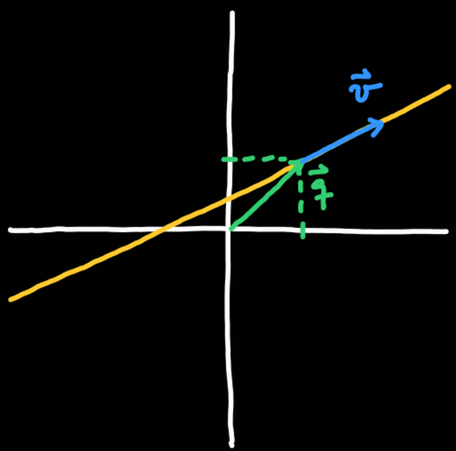
<u>Example:</u> The vector $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ determines a point. The vector $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ determines a point.

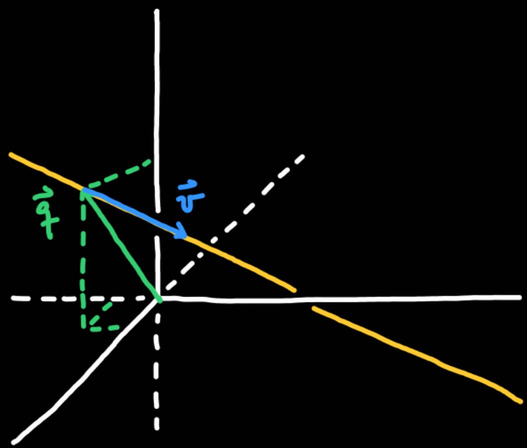




There are two equivalent ways of defining a line or a plane, called the <u>parametric form</u> and the <u>equation form</u>. We can think of the parametric form as allowing degrees of freedom of movement. We can think of the equation form as restricting degrees of freedom of movement.

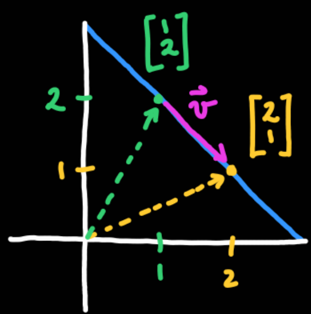
① Parametric form for a line: it is given by a point $\vec{q}$ and a vector $\vec{v}$.





The parametric form is: $\vec{x} = \vec{q} + s\vec{v}$. The point $\vec{q}$ needs to be in the line, and it tells us how the line is translated with respect to the origin. The vector $\vec{v}$ needs to be the direction of movement, we only need one because a line has one degree of freedom. The scalar s ranges through all numbers.

<u>Example</u>: Find a parametric form for the line containing the points $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$.

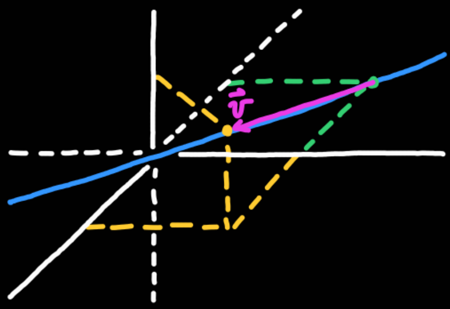


A point in the line is $\vec{q} = \begin{bmatrix}1\\2\end{bmatrix}$. The vector pointing from $\begin{bmatrix}1\\2\end{bmatrix}$ to $\begin{bmatrix}2\\1\end{bmatrix}$ is the direction of movement, so

$\vec{v} = \begin{bmatrix}2\\1\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$. A parametric form for the line is:

$$\vec{x} = \vec{q} + s\vec{v} = \begin{bmatrix}1\\2\end{bmatrix} + s\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1+s\\2-s\end{bmatrix}.$$

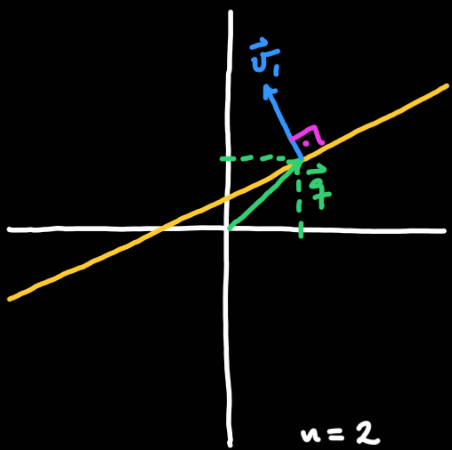
Example: Find a parametric form for the line containing the points $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\0\end{bmatrix}$.

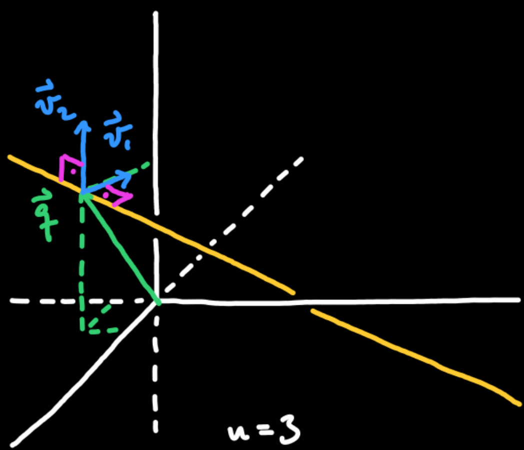


A point in the line is $\vec{q}=\begin{bmatrix}-1\\1\\0\end{bmatrix}$. The vector pointing from $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ to $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is $\vec{v}=\begin{bmatrix}1\\1\\1\end{bmatrix}-\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}2\\0\\1\end{bmatrix}$.

Then: $\vec{x}=\vec{q}+s\vec{v}=\begin{bmatrix}-1\\1\\0\end{bmatrix}+s\begin{bmatrix}2\\0\\1\end{bmatrix}=\begin{bmatrix}-1+2s\\1\\s\end{bmatrix}$.

② Equation form for a line: it is given by a point $\vec{q}$ and vectors $\vec{v}_1,\dots,\vec{v}_{n-1}$ where n is the dimension of the ambient space (a plane is two-dimensional so $n=2$, the world is three-dimensional so $n=3$).



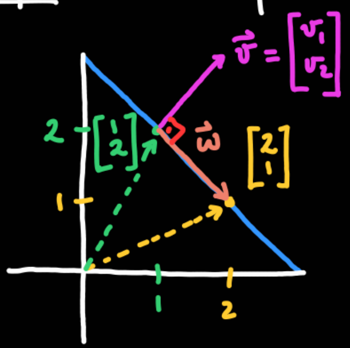


As before, the point $\vec{q}$ needs to be in the line, and it tells us how the line is translated with respect to the origin. The vectors $\vec{v}_1,\dots,\vec{v}_{n-1}$ need to be orthogonal to the direction of movement, each of them gives a direction in which the line is <u>not</u> allowed to point. An ambient space of n dimensions has n degrees of freedom, so to restrict movement down to one degree of freedom we need

$n-1$ forbidden directions, explaining the need for $n-1$ vectors. Each vector gives one equation that must be satisfied, so the equation form is:

$$(\vec{x}-\vec{q})\cdot\vec{v}_1=0$$
$$\vdots$$
$$(\vec{x}-\vec{q})\cdot\vec{v}_{n-1}=0.$$

<u>Example:</u> Find an equation form of the line containing the points $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$.



A point in the line is $\vec{q}=\begin{bmatrix}1\\2\end{bmatrix}$. To find a vector $\vec{v}$ orthogonal to the direction of movement, we first find the direction of movement: $\vec{w}=\begin{bmatrix}2\\1\end{bmatrix}-\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$. Now

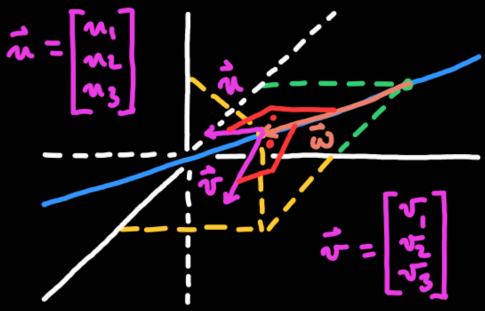
$\vec{v}$ satisfies: $0=\vec{v}\cdot\vec{w}=1\cdot v_1+(-1)\cdot v_2$ so $0=v_1-v_2$ so $v_2=v_1$.

We have infinitely many solutions, one for each non-zero number v_1, but let's make things easy for us and choose $v_1=1$ so $v_2=1$ and $v=\begin{bmatrix}1\\1\end{bmatrix}$. An equation form for the line is then:

$$0=(\vec{x}-\vec{q})\cdot\vec{v}=\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}-\begin{bmatrix}1\\2\end{bmatrix}\right)\cdot\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}x_1-1\\x_2-2\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}=x_1-1+x_2-2=x_1+x_2-3.$$

<u>Example:</u> Find an equation form for the line containing the points $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\0\end{bmatrix}$.

A point in the line is $\vec{q}=\begin{bmatrix}-1\\1\\0\end{bmatrix}$. To find vectors $\vec{u}$ and $\vec{w}$ orthogonal to the



direction of movement, we first find the direction of movement: $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Now $\vec{u}$ and $\vec{v}$ satisfy:

$$0 = \vec{u} \cdot \vec{w} = 2 \cdot u_1 + 0 \cdot u_2 + 1 \cdot u_3 \quad \text{so} \quad u_3 = -2u_1 \quad \text{and} \quad u_2 \text{ is free,}$$

$$0 = \vec{v} \cdot \vec{w} = 2 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3 \quad \text{so} \quad v_3 = -2v_1 \quad \text{and} \quad v_2 \text{ is free.}$$

Let's find $\vec{u}$ first, and since again we have infinitely many solutions, to make life easy let's choose $u_1 = 0$ and $u_2 = -1$ so $u_3 = 0$ and $\vec{u} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$. We cannot choose all $u_1 = u_2 = u_3 = 0$ because we need $\vec{u}$ to be a non-zero vector. Now, as long as $\vec{v}$ is <u>not</u> pointing in the same direction as $\vec{u}$, then $\vec{u}$ and $\vec{v}$ will impose two distinct restrictions to the movement on the line, as we want. If $\vec{v}$ points in the same direction as $\vec{u}$, then they impose the same restriction, which is <u>not</u> what we want. Just to be safe, it is good practice to take $\vec{v}$ orthogonal to $\vec{u}$ (so $\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{w}$), so $\vec{v}$ satisfies:

$$0 = \vec{v} \cdot \vec{u} = 0 \cdot v_1 + (-1) \cdot v_2 + 0 \cdot v_3 \quad \text{so} \quad v_2 = 0.$$

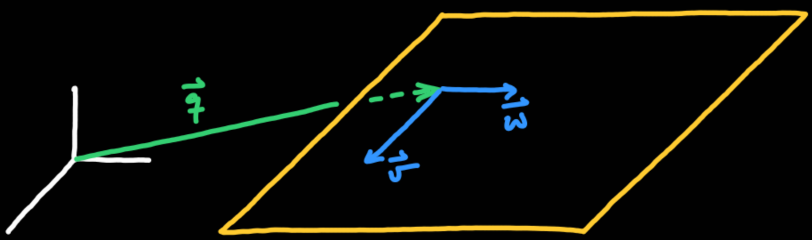
Then $\vec{v}$ satisfies $v_3 = -2v_1$ and $v_2 = 0$, as per usual we have infinitely many solutions, and to make life easy we choose $v_1 = 1$ so $v_3 = -2$ and

$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$. An equation form for the line is then:

$$0 = (\vec{x} - \vec{q}) \cdot \vec{n} = \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1+1 \\ x_2-1 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -x_2 + 1$$

$$0 = (\vec{x} - \vec{q}) \cdot \vec{v} = \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} x_1+1 \\ x_2-1 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = x_1 - 2x_3 + 1.$$

③ Parametric form for a plane: it is given by a point $\vec{q}$ and two vectors $\vec{v}_1$ and $\vec{v}_2$. The parametric form is: $\vec{x} = \vec{q} + s\vec{v} + t\vec{w}$.



The point $\vec{q}$ needs to be in the plane, the vectors $\vec{v}$ and $\vec{w}$ need to be the direction of movement, we need two because a plane has two degrees of freedom.

The scalars s and t range over all numbers.

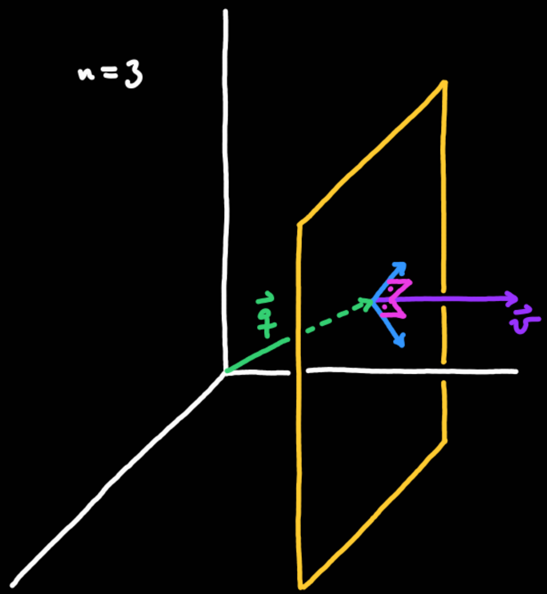
<u>Example:</u> Find a parametric form for the plane containing the points $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

A point in the plane is $\vec{q} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Two vectors in the plane pointing in different directions are the ones pointing from $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and from

$\begin{bmatrix}1\\0\\0\end{bmatrix}$ to $\begin{bmatrix}0\\0\\1\end{bmatrix}$: $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\0\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$.

Then: $\vec{x} = \vec{q} + s\vec{v} + t\vec{w} = \begin{bmatrix}1\\0\\0\end{bmatrix} + s\begin{bmatrix}-1\\1\\0\end{bmatrix} + t\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}1-s-t\\s\\t\end{bmatrix}$.

4 Equation form for a plane: it is given by a point $\vec{q}$ and vectors $\vec{v}_1, \dots, \vec{v}_{n-2}$ where n is the dimension of the ambient space (the world is three-dimensional so $n=3$).



The point $\vec{q}$ needs to be in the plane, and the vectors $\vec{v}_1, \dots, \vec{v}_{n-2}$ need to be orthogonal to the plane. An ambient space of n dimensions has n degrees of freedom, so to restrict movement down to two degrees of freedom we need $n-2$ forbidden directions, explaining the need for $n-2$ vectors. Each vector gives one equation that must be satisfied, so the equation form is:

$$(\vec{x}-\vec{q})\cdot\vec{v}_1 = 0$$
$$\vdots$$
$$(\vec{x}-\vec{q})\cdot\vec{v}_{n-2} = 0.$$

Example: Find an equation form for the plane containing the points $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\begin{bmatrix}0\\0\\1\end{bmatrix}$.

A point in the plane is $\vec{q} = \begin{bmatrix}1\\0\\0\end{bmatrix}$. To find a vector orthogonal to the plane, we first find two vectors in the plane (that are not on the same line):

$$\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\0\end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix},$$

and we take their cross product:

$$\vec{n} = \vec{v} \times \vec{w} = \det\begin{bmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix} = 1\cdot\vec{e}_1 - (-1)\cdot\vec{e}_2 + 1\cdot\vec{e}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

An equation form for the plane is then:

$$0 = (\vec{x} - \vec{q})\cdot\vec{n} = \left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\right)\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}x_1-1\\x_2\\x_3\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} = x_1 + x_2 + x_3 - 1.$$