2.6. Introduction to linear systems.

A <u>linear system of equations</u> is a collection of equations where the highest degree of the variables is one.

<u>Example:</u> The following is a linear system of three equations in three variables.

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases}$$

Equivalently, a linear system of equations is an equation form of a geometric figure. Each equation in the linear system is imposing a restriction, but these can be redundant. Solving a linear system is giving a parametric form of the geometric figure being described.

<u>Example:</u> Determine whether the following system of equations describes a point, a line, or a plane.

(i) $$\begin{cases} x + 2y = 1 \\ 3x + 4y = 1 \end{cases}$$

(iii) $$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases}$$

(i) The system has two variables, so the ambient space is a plane, which has two degrees of freedom. The first equation imposes the restriction of being orthogonal to the vector $\begin{bmatrix}1\\2\end{bmatrix}$, and the second equation imposes the restriction of being orthogonal to the vector $\begin{bmatrix}3\\4\end{bmatrix}$. Since $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}$ are different directions, we have two distinct restrictions. Being in a plane, we are left with <u>no</u> degrees of freedom, so the system describes a point.

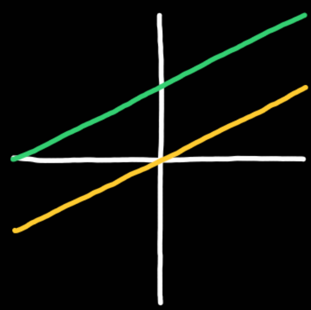
(iii) The system has three variables, so the ambient space has three degrees of freedom. The first equation imposes being orthogonal to $\begin{bmatrix}1\\2\\3\end{bmatrix}$, the second equation imposes being orthogonal to $\begin{bmatrix}4\\5\\6\end{bmatrix}$, and the third equation imposes being orthogonal to $\begin{bmatrix}7\\8\\9\end{bmatrix}$. Since $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}4\\5\\6\end{bmatrix}$ are different directions, the first two equations impose different restrictions. Since $\begin{bmatrix}7\\8\\9\end{bmatrix} = (-1)\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} + 2\cdot\begin{bmatrix}4\\5\\6\end{bmatrix}$, the third equation is <u>not</u> imposing an additional restriction: a direction orthogonal to $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}4\\5\\6\end{bmatrix}$ is automatically orthogonal to $\begin{bmatrix}7\\8\\9\end{bmatrix}$. We begin with three degrees of freedom, so after imposing two restrictions we are left with one degree of freedom. This is <u>not</u> quite enough to declare that the system determines a line, but since the options we are given are point, line, and plane, we indeed must have a line.

A linear system of equations can have one solution, infinitely many solutions, or <u>no</u> solutions.

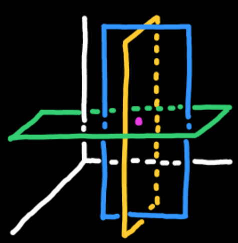
Geometrically, each equation of a linear system determines a geometric figure, and the solution of a system of equations is the intersection of these geometric figures. Having one solution corresponds to the figures intersecting at a point, having infinitely many solutions corresponds to the figures intersecting along at least one degree of freedom, and having <u>no</u> solutions corresponds to the figures <u>not</u> intersecting.

one solution
intersection at a point.

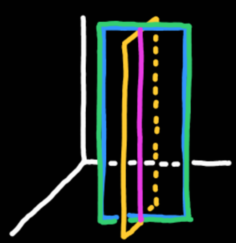
infinite solutions
intersection at a line.



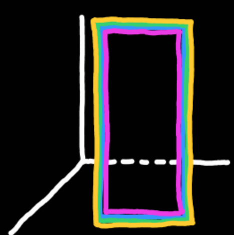
<u>no</u> solution
<u>no</u> intersection.



one solution
point intersection.



infinite solutions
line intersection.



infinite solutions
plane intersection.

<u>no</u> solution
<u>no</u> intersection.

The determinant can be used to figure out if a system of equations has a unique solution. Given a system of equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = c_1 \\ a_{21}x_1 + a_{22}x_2 = c_2 \end{cases}$$

it has one solution exactly when $\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \neq 0$. Given a system of equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3 \end{cases}$$

it has one solution exactly when $\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \neq 0$. In general, the system:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = c_1 \\ \quad\vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = c_n \end{cases}$$

has one solution exactly when $\det\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \neq 0$. When this determinant is zero, it conveys a redundancy on the left hand side of the system of equations. The formal way of describing such a redundancy is via the notion of linear dependence, and having <u>no</u> redundancy is conveyed through linear independence. To introduce these, it is also useful to describe what a linear combination of vectors is, as well as the concept of the span of vectors. We now do all of this.

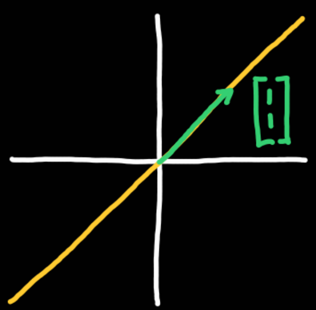
A <u>linear combination</u> of the vectors $\vec{v}_1, \dots, \vec{v}_n$ is the vector $s_1\vec{v}_1 + \dots + s_n\vec{v}_n$

where $s_1, \dots, s_n$ are scalars.

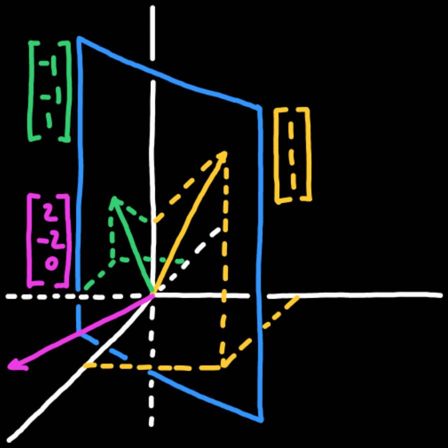
<u>Example:</u> The vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ because $\begin{bmatrix}1\\2\\3\end{bmatrix} = 1\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}$. The vector $\begin{bmatrix}-4\\-5\\5\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}-1\\0\\3\end{bmatrix}$ and $\begin{bmatrix}2\\5\\1\end{bmatrix}$ because $\begin{bmatrix}-4\\-5\\5\end{bmatrix} = 2\cdot\begin{bmatrix}-1\\0\\3\end{bmatrix} + (-1)\cdot\begin{bmatrix}2\\5\\1\end{bmatrix}$. Any vector $\vec{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$ is a linear combination of the standard basis vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$.

The <u>span</u> of the vectors $\vec{v}_1, \dots, \vec{v}_n$ is the collection of all linear combinations of the vectors $\vec{v}_1, \dots, \vec{v}_n$.

<u>Example:</u> The span of the vector $\begin{bmatrix}1\\1\end{bmatrix}$ can be understood as the line going through the origin in the direction of $\begin{bmatrix}1\\1\end{bmatrix}$.



The span of the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\-1\\1\end{bmatrix}$ can be understood as the plane going through the origin and perpendicular to $\begin{bmatrix}2\\-2\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} \times \begin{bmatrix}-1\\-1\\1\end{bmatrix}$.



The span of the vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is all the two dimensional plane.

The span of the vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is all the three dimensional space. The span of the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is all the three dimensional space.

The vectors $\vec{v}_1, \dots, \vec{v}_n$ are said to be <u>linearly dependent</u> if there are scalars $s_1, \dots, s_n$, at least one of them non-zero, such that $s_1\vec{v}_1 + \cdots + s_n\vec{v}_n = \vec{0}$.

<u>Example:</u> The vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly dependent because $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1)\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-2)\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-3)\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The vectors $\begin{bmatrix} -4 \\ -5 \\ 5 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ are linearly dependent because $\begin{bmatrix} -4 \\ -5 \\ 5 \end{bmatrix} + (-2)\cdot\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + 1\cdot\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

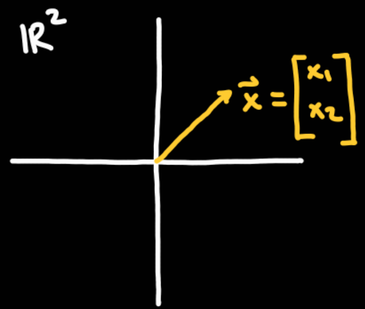
<u>Example:</u> Two linearly dependent vectors lie in the same line. Three linearly dependent vectors lie in the same plane. Four linearly dependent vectors lie in the same three dimensional space.

A collection of vectors is linearly dependent exactly when (at least) one of the vectors is a linear combination of the others.

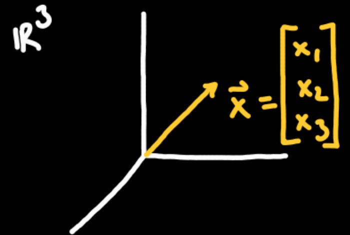
The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are said to be <u>linearly independent</u> when they are <u>not</u> linearly dependent. Namely, if there are scalars $s_1, \ldots, s_n$ such that $s_1\vec{v}_1 + \cdots + s_n\vec{v}_n = \vec{0}$, then $s_1 = \cdots = s_n = 0$, all the scalars must be zero.

<u>Example:</u> The vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$ are linearly independent. The vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\-1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\-1\\0\end{bmatrix}$ are linearly independent. The vectors $\begin{bmatrix}-1\\0\\3\end{bmatrix}$ and $\begin{bmatrix}2\\5\\1\end{bmatrix}$ are linearly independent.

We denote by $\mathbb{R}^n$ the collection of all vectors with n entries. That is, $\mathbb{R}^2$ is the usual plane with points seen as vectors with source the origin,



and $\mathbb{R}^3$ is the usual three dimensional space with points seen as vectors with source the origin.



A collection of linearly independent vectors in $\mathbb{R}^n$ has at most n vectors.

Equivalently, a collection of more than n vectors in $\mathbb{R}^n$ is linearly dependent.

This is the sense in which we can understand the number of linearly independent vectors as the degrees of freedom: each one of those linearly independent vector is allowing movement in the direction in which it points. A vector that is linearly dependent on others does not allow movement in a new direction, because we could already move in that direction by doing a linear combination of the previous vectors.

Example: The vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\-1\\1\end{bmatrix}$ are linearly independent. The plane with parametric form $\vec{x} = \vec{q} + s\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} + t\cdot\begin{bmatrix}-1\\-1\\1\end{bmatrix}$ has then two degrees of freedom. The vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\-1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\1\\0\end{bmatrix}$ are linearly dependent because $\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}-1\\-1\\1\end{bmatrix} - 2\cdot\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.
The plane with parametric form $\vec{x} = \vec{q} + r\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} + s\cdot\begin{bmatrix}-1\\-1\\1\end{bmatrix} + t\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}$ has then two degrees of freedom because writing $\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{1}{2}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{2}\cdot\begin{bmatrix}-1\\-1\\1\end{bmatrix}$ can be used to give the equivalent parametric form

$$\vec{x} = \vec{q} + r\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} + s\cdot\begin{bmatrix}-1\\-1\\1\end{bmatrix} + t\cdot\begin{bmatrix}1\\1\\0\end{bmatrix} = \vec{q} + r\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} + s\cdot\begin{bmatrix}-1\\-1\\1\end{bmatrix} + t\cdot\left(\frac{1}{2}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{2}\cdot\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right) =$$

$$= \vec{q} + \left(r + \frac{t}{2}\right) \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \left(s - \frac{t}{2}\right) \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

The vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ are linearly independent. The space with parametric form $\vec{x} = \vec{q} + r \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ has then three degrees of freedom. As we just saw, the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ are linearly dependent because $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, so the space with parametric form

$$\vec{x} = \vec{q} + a \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \vec{q} + \left(a + \frac{c}{2}\right) \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \left(b - \frac{c}{2}\right) \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + d \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

has then three degrees of freedom.

A collection of n linearly independent vectors in $\mathbb{R}^n$ is called a <u>basis</u>. Having $\vec{v}_1, \dots, \vec{v}_n$ a basis of $\mathbb{R}^n$ means that every point $\vec{x}$ in $\mathbb{R}^n$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_n$.

<u>Example:</u> The vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are a basis of $\mathbb{R}^2$. The vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are a basis of $\mathbb{R}^2$. The vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ are <u>not</u> a basis of $\mathbb{R}^3$. The vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ are a basis of $\mathbb{R}^3$. The vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are a basis of $\mathbb{R}^3$.

The <u>dimension</u> of the span of the linearly independent vectors $\vec{v}_1, \dots, \vec{v}_n$ is n.

3. Solving linear systems

3.1. Linear systems

A general linear system of m equations in n variables has the form:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = c_1 \\ \qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = c_n \end{cases}$$

where the coefficients a_{ij} and c_i are known for all i and all j. The solution to the system are all the values $x_1, \dots, x_n$ that satisfy all the equations.

<u>Example:</u> A linear system of 4 equations and 5 variables is:

$$\begin{cases} x_1 + 0\cdot x_2 + 2x_3 + x_4 + 0\cdot x_5 = 1 \\ 0\cdot x_1 + 1\cdot x_2 + 2x_3 + x_4 + x_5 = 1 \\ x_1 + x_2 + 2x_3 + 0\cdot x_4 + x_5 = 0 \\ 0\cdot x_1 + x_2 + 0\cdot x_3 + x_4 + 2x_5 = -1. \end{cases}$$

A linear system of 3 equations and 2 variables is:

$$\begin{cases} x_1 + x_2 = 0 \\ 2x_1 + x_2 = 1 \\ 3x_1 + 2x_2 = 1. \end{cases}$$

To solve a system of equations (that is, to find the parametric form corresponding to the given equation form) we will do some operations on the equations that do <u>not</u>

change the solutions but makes the system simpler. These three operations are:

(i) Multiply an equation by a non-zero scalar,

(ii) Add an equation to another equation,

(iii) Interchange two equations,

and are known as <u>elementary row operations</u>.

<u>Example</u>: Consider the system:

$$\begin{cases} x_1 + 2x_3 + x_4 = 1 \\ x_2 + 2x_3 + x_4 + x_5 = 1 \\ x_1 + x_2 + 2x_3 + x_5 = 0 \\ x_2 + x_4 + 2x_5 = -1. \end{cases}$$

We can multiply the first equation by -2, and obtain an equivalent system of equations with the same solutions:

$$\begin{cases} -2x_1 - 4x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + x_4 + x_5 = 1 \\ x_1 + x_2 + 2x_3 + x_5 = 0 \\ x_2 + x_4 + 2x_5 = -1. \end{cases}$$

We can add the fourth equation to the second equation, and obtain an equivalent system of equations with the same solutions:

$$\begin{cases} -2x_1 - 4x_3 - 2x_4 = -2 \\ 2x_2 + 2x_3 + 2x_4 + 3x_5 = 0 \\ x_1 + x_2 + 2x_3 + x_5 = 0 \\ x_2 + x_4 + 2x_5 = -1. \end{cases}$$

We can interchange the third and fourth equations, and obtain an equivalent system of equations with the same solutions:

$$\begin{cases} -2x_1 - 4x_3 - 2x_4 = -2 \\ 2x_2 + 2x_3 + 2x_4 + 3x_5 = 0 \\ x_2 + x_4 + 2x_5 = -1 \\ x_1 + x_2 + 2x_3 + x_5 = 0. \end{cases}$$

A convenient way of writing a system of equations is to set up its <u>augmented matrix.</u> The augmented matrix keeps track of the coefficients in a systematic way. The augmented matrix of the general system:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = c_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = c_n \end{cases} \quad \text{is} \quad \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & c_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & c_n \end{array}\right].$$

<u>Example:</u> The following system and augmented matrix contain the same information.

$$\begin{cases} x_1 + 2x_3 + x_4 = 1 \\ x_2 + 2x_3 + x_4 + x_5 = 1 \\ x_1 + x_2 + 2x_3 + x_5 = 0 \\ x_2 + x_4 + 2x_5 = -1 \end{cases} \quad \text{corresponds to} \quad \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \end{array}\right]$$

The three allowed operations on the equations of the system can now be done on the rows of the corresponding augmented matrix.

<u>Example:</u> Consider the following system and its corresponding augmented matrix.

$$\begin{cases} x_1 + 2x_3 + x_4 = 1 \\ x_2 + 2x_3 + x_4 + x_5 = 1 \\ x_1 + x_2 + 2x_3 + x_5 = 0 \\ x_2 + x_4 + 2x_5 = -1 \end{cases} \qquad \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$(-2)\cdot(\text{equation } 1) \qquad \downarrow \qquad (-2)\cdot(\text{row } 1)$$

$$\begin{cases} -2x_1 - 4x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + x_4 + x_5 = 1 \\ x_1 + x_2 + 2x_3 + x_5 = 0 \\ x_2 + x_4 + 2x_5 = -1. \end{cases} \qquad \left[\begin{array}{ccccc|c} -2 & 0 & -4 & -2 & 0 & -2 \\ 0 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$(\text{equation } 2) + (\text{equation } 4) \qquad \downarrow \qquad (\text{row } 2) + (\text{row } 4)$$

$$\begin{cases} -2x_1 - 4x_3 - 2x_4 = -2 \\ 2x_2 + 2x_3 + 2x_4 + 3x_5 = 0 \\ x_1 + x_2 + 2x_3 + x_5 = 0 \\ x_2 + x_4 + 2x_5 = -1. \end{cases} \qquad \left[\begin{array}{ccccc|c} -2 & 0 & -4 & -2 & 0 & -2 \\ 0 & 2 & 2 & 2 & 3 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$(\text{equation } 3) \leftrightarrow (\text{equation } 4) \qquad \downarrow \qquad (\text{row } 3) \leftrightarrow (\text{row } 4)$$

$$\begin{cases} -2x_1 - 4x_3 - 2x_4 = -2 \\ 2x_2 + 2x_3 + 2x_4 + 3x_5 = 0 \\ x_2 + x_4 + 2x_5 = -1 \\ x_1 + x_2 + 2x_3 + x_5 = 0. \end{cases} \qquad \left[\begin{array}{ccccc|c} -2 & 0 & -4 & -2 & 0 & -2 \\ 0 & 2 & 2 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 1 & 1 & 2 & 0 & 1 & 0 \end{array}\right]$$

Note how the system and the augmented matrix after an operation continue to correspond to each other.

We now want to systematically use these elementary row operations to simplify the augmented matrix of a system.