## 3.2. Gaussian elimination

The three elementary row operations allow us to transform a system of equations into a different system of equations with the same solutions. We will use these elementary row operations to systematically simplify a system of equations until we can directly read off the solutions. This process is called <u>Gaussian elimination</u>.

<u>Example:</u> Solve the following linear system of equations.

$$\begin{cases} 2x_1 + 8x_2 + 4x_3 = 2 \\ 2x_1 + 5x_2 + x_3 = 5 \\ 4x_1 + 10x_2 - x_3 = 1 \end{cases}$$

To do this, we first write the corresponding augmented matrix, and we do row operations until the left side of the augmented matrix has ones in the diagonal and zeros everywhere else.

$$\left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}\cdot R_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right] \xrightarrow[R_3 - 4\cdot R_1]{R_2 - 2\cdot R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -9 & -3 \end{array}\right] \xrightarrow{\frac{1}{-3}\cdot R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -6 & -9 & -7 \end{array}\right] \xrightarrow{R_3 + 6\cdot R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{array}\right] \xrightarrow{\frac{1}{-3}\cdot R_3} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 - 4\cdot R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_2 - R_1]{R_1 + 2\cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

We can now translate the augmented matrix back to a system of linear equations, obtaining the following.

$$\begin{cases} x_1 = 11 \\ x_2 = -4 \\ x_3 = 3 \end{cases}$$

This is indeed a solved system of equations, with $\begin{bmatrix} 11 \\ -4 \\ 3 \end{bmatrix}$ the only solution.

The final form of the augmented matrix after doing as many simplifications as possible is known as <u>reduced row echelon form</u>. In general, it will have the following shape.

$$\left[\begin{array}{cccccccccc|c} 1 & * & * & 0 & * & 0 & 0 & * & * & * & * \\ & & & 1 & * & 0 & 0 & * & * & * & * \\ & & & & & 1 & 0 & * & * & * & * \\ & & & & & & 1 & * & * & * & * \\ & & & & & & & & & & * \\ & & & & & & & & & & * \end{array}\right]$$

In the shape above there must be zeros under the horizontal line. Note that in the example above we could have stopped after obtaining the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

because the corresponding linear system of equations is

$$\begin{cases} x_1 + 4x_2 + 2x_3 = 1 \\ x_2 + x_3 = -1 \\ x_3 = 3 \end{cases}$$

which can be solved by $x_3 = 3$ from the third equation, substituting it into the second equation to obtain $x_2 = -1 - x_3 = -1 - 3 = -4$, and substitute it into the first equation to obtain $x_1 = 1 - 4x_2 - 2x_3 = 1 + 16 - 6 = 11$. This intermediate form before fully simplifying is known as <u>row echelon form</u>. In general, it will have the following shape, where again there must be zeros under the horizontal line.

$$\left[\begin{array}{cccccccccc|c} 1 & * & * & * & * & * & * & * & * & * & * \\ & & & 1 & * & * & * & * & * & * & * \\ & & & & & 1 & * & * & * & * & * \\ & & & & & & 1 & * & * & * & * \\ & & & & & & & & & & * \\ & & & & & & & & & & * \end{array}\right]$$

It is <u>not</u> always possible to obtain the ideal row reduced echelon form. This could potentially indicate that we have infinitely many solutions, or no solutions at all.

Example: Solve the following linear system of equations.

$$\begin{cases} 2x_1 - 4x_2 + 6x_3 = 4 \\ 4x_1 - 2x_2 = 2 \\ -6x_1 + 6x_3 = -2 \end{cases}$$

Proceeding systematically, we will again attempt to obtain an augmented matrix whose left side has ones in the diagonal and zeros everywhere else.

$$\left[\begin{array}{ccc|c} 2 & -4 & 6 & 4 \\ 4 & -2 & 0 & 2 \\ -6 & 0 & 6 & -2 \end{array}\right] \xrightarrow{\frac{1}{2}\cdot R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 4 & -2 & 0 & 2 \\ -6 & 0 & 6 & -2 \end{array}\right] \xrightarrow[R_3+6\cdot R_1]{R_2-4\cdot R_1}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 6 & -12 & -6 \\ 0 & -12 & 24 & 10 \end{array}\right] \xrightarrow{\frac{1}{6}\cdot R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & -12 & 24 & 10 \end{array}\right] \xrightarrow{R_3+12\cdot R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right] \xrightarrow{\frac{1}{-2}\cdot R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1+2R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The very last step is not necessary, but it is good practice to make all entries above the "diagonal" ones zero. Translating the augmented matrix back to a system of linear equations gives the following.

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - 2x_3 = 0 \\ 0 = 1 \end{cases}$$

Since the equation $0=1$ never holds true, the system has <u>no</u> solutions.

<u>Example:</u> Solve the following linear system of equations.

$$\begin{cases} -2x_1 - 2x_2 + 4x_3 = 10 \\ 5x_1 + x_2 - 2x_3 = 7 \\ x_1 - 3x_2 + 6x_3 = 27 \end{cases}$$

We again attempt to obtain an augmented matrix whose left side has ones in the diagonal and zeros everywhere else.

$$\left[\begin{array}{ccc|c} -2 & -2 & 4 & 10 \\ 5 & 1 & -2 & 7 \\ 1 & -3 & 6 & 27 \end{array}\right] \xrightarrow{\frac{1}{-2}\cdot R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -5 \\ 5 & 1 & -2 & 7 \\ 1 & -3 & 6 & 27 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 5\cdot R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -5 \\ 0 & -4 & 8 & 32 \\ 0 & -4 & 8 & 32 \end{array}\right] \xrightarrow[\frac{1}{-4}\cdot R_3]{\frac{1}{-4}\cdot R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -5 \\ 0 & 1 & -2 & -8 \\ 0 & 1 & -2 & -8 \end{array}\right] \xrightarrow[R_3 - R_2]{R_1 - R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This augmented matrix translates to the following system of linear equations.

$$\begin{cases} x_1 = 3 \\ x_2 - 2x_3 = -8 \\ 0 = 0 \end{cases}$$

The third equation is irrelevant, since it provides <u>no</u> restrictions and <u>no</u> additional information at all, it is simply a mathematical truth. The first equation gives $x_1 = 3$. The second imposes a dependency between $x_2$ and $x_3$, which we can write as either $x_2 = 2x_3 - 8$ or $x_3 = \frac{1}{2} \cdot x_2 + 4$. To make our life simpler, we avoid fractions and choose $x_2 = 2x_3 - 8$. We then have the following system of linear equations.

$$\begin{cases} x_1 = 3 \\ x_2 = 2x_3 - 8 \end{cases}$$

Since $x_3$ is <u>not</u> constrained, we set $x_3 = s$ a parameter, obtaining

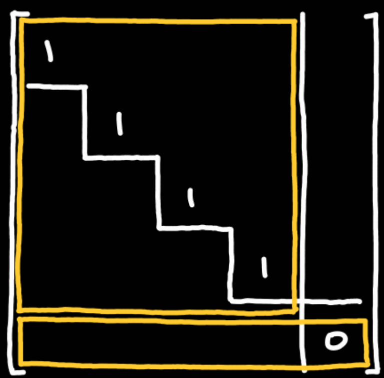
$$\begin{cases} x_1 = 3 \\ x_2 = -8 + 2s \\ x_3 = s \end{cases}$$

which can also be written in the following parametric form.

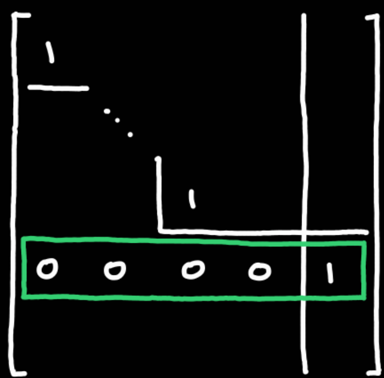
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

This gives infinitely many solutions, namely all the points in the line given by the parametric form above.

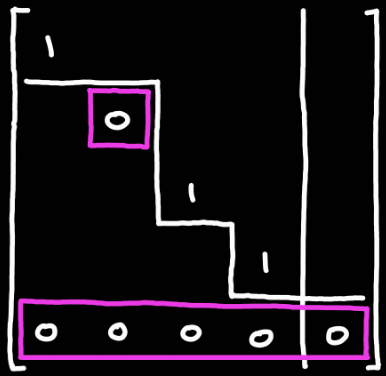
In general, there are three possible reduced row echelon forms, each of them corresponding to whether the original system of equations has exactly one solution, no solutions, or infinitely many solutions.



If the reduced row echelon form has ones in the diagonal and all other rows are zero, the original system has exactly one solution.



If the reduced row echelon form has a row with a one in the last column and zeros in all other columns, the original system has no solutions.



If the reduced row echelon form does not have all ones in the diagonal and does not have a row with a one in the last column and zeros in all other columns, the original system has infinitely many solutions.

To make this concept precise we use the following notion. The <u>rank of a matrix</u> is the number of non-zero rows in its reduced row echelon form.

<u>Example:</u> Find the rank of the following augmented matrices.

(i) The rank of

$$\left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right] \quad \text{is } 3 \text{ because} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

is its reduced row echelon form, which has three non-zero rows.

(ii) The rank of

$$\left[\begin{array}{ccc|c} 2 & -4 & 6 & 4 \\ 4 & -2 & 0 & 2 \\ -6 & 0 & 6 & -2 \end{array}\right] \quad \text{is } 3 \text{ because} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

is its reduced row echelon form, which has three non-zero rows.

(iii) The rank of

$$\left[\begin{array}{ccc|c} -2 & -2 & 4 & 10 \\ 5 & 1 & -2 & 7 \\ 1 & -3 & 6 & 27 \end{array}\right] \quad \text{is 2 because} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

is its reduced row echelon form, which has two non-zero rows.

The existence and number of solutions can now be determined in terms of the rank of a matrix and the number of unknowns.

(a) If the rank of the augmented matrix equals the number of unknowns, the system has exactly one solution.

(b) If the rank of the augmented matrix is strictly greater than the rank of the matrix obtained by removing the last column of the augmented matrix, the system has <u>no</u> solutions.

(c) If the rank of the augmented matrix equals the rank of the matrix obtained by removing the last column of the augmented matrix and is strictly smaller than the number of unknowns, the system has infinitely many solutions.

<u>Example:</u> Determine the number of solutions of the following systems using ranks.

(i) The system

$$\begin{cases} 2x_1 + 8x_2 + 4x_3 = 2 \\ 2x_1 + 5x_2 + x_3 = 5 \\ 4x_1 + 10x_2 - x_3 = 1 \end{cases} \quad \text{with augmented matrix} \quad \left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right]$$

has exactly one solution because it has three unknowns and the rank of the augmented matrix is three.

(ii) The system

$$\begin{cases} 2x_1 - 4x_2 + 6x_3 = 4 \\ 4x_1 - 2x_2 = 2 \\ -6x_1 + 6x_3 = -2 \end{cases} \quad \text{with augmented matrix} \quad \left[\begin{array}{ccc|c} 2 & -4 & 6 & 4 \\ 4 & -2 & 0 & 2 \\ -6 & 0 & 6 & -2 \end{array}\right]$$

has <u>no</u> solutions because the rank of the augmented matrix is three and the rank of the matrix obtained by removing the last column of the augmented matrix is two:

$$\begin{bmatrix} 2 & -4 & 6 \\ 4 & -2 & 0 \\ -6 & 0 & 6 \end{bmatrix} \quad \text{has reduced row echelon form} \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

which indeed has two non-zero rows.

(iii) The system

$$\begin{cases} -2x_1 - 2x_2 + 4x_3 = 10 \\ 5x_1 + x_2 - 2x_3 = 7 \\ x_1 - 3x_2 + 6x_3 = 27 \end{cases}$$

has infinitely many solutions because the number of unknowns is three, the rank of the augmented matrix is two, and the rank of the matrix obtained by removing the last column of the augmented matrix is two;

$$\begin{bmatrix} -2 & -2 & 4 \\ 5 & 1 & -2 \\ 1 & -3 & 6 \end{bmatrix} \text{ has reduced row echelon form } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

which indeed has two non-zero rows.

## 3.3. Homogeneous equations.

A linear system of equations is said to be <u>homogeneous</u> when it has the form

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0, \end{cases}$$

that is, all scalars <u>not</u> multiplying a variable are zero. Homogeneous systems of equations describe geometric figures that go through the origin. This is because $\vec{x} = \vec{0}$, namely $x_1 = \cdots = x_n = 0$, is always a solution of the system. In particular, a homogeneous system always has at least one solution (so they have exactly one solution or infinitely many solutions). Homogeneous systems appear when we are figuring out whether a collection of vectors is linearly dependent or linearly independent.

<u>Example:</u> Determine if the following collections of vectors are linearly dependent or linearly independent.

(i) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}.$

(ii) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}.$

(iii) $\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}.$

(i) The vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$, and $\begin{bmatrix}7\\8\\9\end{bmatrix}$ are linearly dependent when there are scalars $s_1, s_2$, and $s_3$, at least one non-zero, satisfying

$$s_1\begin{bmatrix}1\\2\\3\end{bmatrix} + s_2\begin{bmatrix}4\\5\\6\end{bmatrix} + s_3\begin{bmatrix}7\\8\\9\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$, and $\begin{bmatrix}7\\8\\9\end{bmatrix}$ are linearly independent when the only solution to the equation

$$s_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + s_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is $s_1 = s_2 = s_3 = 0$. It then suffices to solve the equation above, because based on its solutions we will be able to determine the linear dependence or linear independence we want. We transform the equation into a system:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = s_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + s_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} s_1 \\ 2s_1 \\ 3s_1 \end{bmatrix} + \begin{bmatrix} 4s_2 \\ 5s_2 \\ 6s_2 \end{bmatrix} + \begin{bmatrix} 7s_3 \\ 8s_3 \\ 9s_3 \end{bmatrix} = \begin{bmatrix} s_1 + 4s_2 + 7s_3 \\ 2s_1 + 5s_2 + 8s_3 \\ 3s_1 + 6s_2 + 9s_3 \end{bmatrix}$$

so we have:

$$\begin{cases} s_1 + 4s_2 + 7s_3 = 0 \\ 2s_1 + 5s_2 + 8s_3 = 0 \\ 3s_1 + 6s_2 + 9s_3 = 0 \end{cases}$$

which we can write in an augmented matrix and simplify as follows.

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \xrightarrow{\frac{1}{-3} \cdot R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \xrightarrow{R_3 + 6 \cdot R_2} \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 4 \cdot R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there are three unknowns, the rank of the augmented matrix is two, and the rank of the non-augmented matrix is two, the system has infinitely many solutions. One solution will be $s_1 = s_2 = s_3 = 0$, and all the others will have at least one of $s_1$, $s_2$, and $s_3$ not zero. This means that the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ are linearly dependent. This finishes the problem as stated, but if we are also asked to explain which vector depends on which other vectors, we need to fully solve the system. The second row of the reduced row echelon form of the augmented matrix gives $s_2 = -2s_3$, the first row gives $s_1 = s_3$, and since $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is always a solution (namely the point $\vec{0}$ is contained in the geometric figure described by the system) we have the parametric form:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} s_3 \\ -2s_3 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

We are usually used to seeing it with the variables named $x_1, x_2, x_3$ and the parameter named $s$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

(note that indeed $x_3 = s$ and above $s_3 = s_3$)

but we should not be intimidated by the unusual appearance. It does not matter how we call the variables, the last two parametric forms are equivalent. To explicitly find a non-zero solution we make our life easy and choose $s_3 = 1$, which gives $s_2 = -2$ and $s_1 = 1$. Indeed:

$$1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 1 \cdot \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or equivalently} \quad \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 2 \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(ii) We could be tempted to do exactly the same process for $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, but since these are the first two columns of the augmented matrix, we can save ourselves some work. We are solving

$$s_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{whose augmented matrix is} \quad \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{array}\right]$$

so looking at the elementary row operations we just did, we obtain that its reduced row echelon form is $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$. There are two unknowns, the augmented matrix has rank two (the non-augmented matrix also has rank two), so there is exactly one solution to the system. Since $s_1 = s_2 = s_3 = 0$ is always a solution for homogeneous systems, it must be the unique solution. The vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ are then linearly independent.

(iii) Let's use the vectors $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to illustrate an expedited way of solving these types of problems. We first set up the system and its augmented matrix.

$$\begin{cases} s_1 + 2s_2 + s_3 = 0 \\ 2s_1 + 2s_2 + s_3 = 0 \\ 2s_1 + s_2 + s_3 = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 1 & 0 \end{array}\right]$$

We simplify the augmented matrix. Sometimes it is easier to start from the bottom row.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_2 - R_3]{R_1 - R_3} \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 - R_2}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{-1}\cdot R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - 2\cdot R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

We use the rank to determine the number of solutions. There are three unknowns and the rank of the augmented matrix is three, so there is exactly one solution, which is $s_1 = s_2 = s_3 = 0$. The vectors $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are then linearly independent.