3.4. Geometric applications.

We now use the tools we have seen in a range of applications.

Example: Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$.

(i) Are $\vec{u}, \vec{v}, \vec{w}$ linearly independent?

(ii) Are $\vec{u}, \vec{v}, \text{proj}_{\vec{v}}\vec{u}$ linearly independent? Justify it using a volume.

(iii) Are $\vec{u}, \vec{v}, \text{proj}_{\vec{w}}\vec{v}$ linearly independent? Justify it without computing $\text{proj}_{\vec{w}}\vec{v}$.

(i) We are solving the equation:

$$s_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} + s_3 \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives the system of equations:

$$\begin{cases} s_1 - s_2 + 2s_3 = 0 \\ 2s_1 + 3s_2 - s_3 = 0 \\ 3s_1 - s_2 - 4s_3 = 0 \end{cases}$$

which can be solved via the following Gaussian elimination.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & 3 & -1 & 0 \\ 3 & -1 & -4 & 0 \end{array}\right] \xrightarrow[R_3 - 3\cdot R_1]{R_2 - 2\cdot R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 2 & -10 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}\cdot R_3]{\frac{1}{5}\cdot R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -5 & 0 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right] \xrightarrow{\frac{1}{-6}\cdot R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

This augmented matrix in row echelon form corresponds to the system:

$$\begin{cases} s_1 - s_2 + 2s_3 = 0 \\ s_2 - s_3 = 0 \\ s_3 = 0 \end{cases}$$

which has the unique solution $s_1 = s_2 = s_3 = 0$. The vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly independent.

(ii) We first compute:

$$\text{proj}_{\vec{v}}\,\vec{u} = (\vec{u}\cdot\vec{v})\cdot\frac{\vec{v}}{\|\vec{v}\|^2} = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}\right)\cdot\frac{1}{1^2+3^2+1^2}\cdot\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \frac{2}{11}\cdot\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2/11 \\ 6/11 \\ -2/11 \end{bmatrix}.$$

The volume of the parallelepiped determined by $\vec{u}, \vec{v}$, and $\text{proj}_{\vec{v}}\,\vec{u}$ is:

$$\left|\det\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & -1 \\ -2/11 & 6/11 & -2/11 \end{bmatrix}\right| = 1\cdot(-6/11 + 6/11) + 2\cdot(2/11 - 2/11) + 3\cdot(-6/11 + 6/11) = 0.$$

This means that $\vec{u}, \vec{v}$, and $\text{proj}_{\vec{v}}\,\vec{u}$ are in the same plane, so they are linearly dependent.

(iii) The vectors $\vec{v}$ and $\vec{w}$ are not orthogonal because:

$$\vec{v}\cdot\vec{w} = (-1)\cdot 2 + 3\cdot(-1) + (-1)\cdot(-4) = -1 \neq 0.$$

Thus $\text{proj}_{\vec{w}}\vec{v} \neq \vec{0}$. Since $\text{proj}_{\vec{w}}\vec{v}$ is not zero and points in the same direction as $\vec{w}$, then the three vectors $\vec{u}, \vec{v}, \text{proj}_{\vec{w}}\vec{v}$ are linearly independent if and only if the three vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly independent. We saw in (i) that $\vec{u}, \vec{v}, \vec{w}$ are linearly independent, so $\vec{u}, \vec{v}, \text{proj}_{\vec{w}}\vec{v}$ are linearly independent.

Example: Find all values k for which the plane $x_1 + \frac{3}{5}x_2 + (k-1)x_3 = 1$ and the plane $\vec{x} = \begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{s}{3}\begin{bmatrix}0\\2\\3\end{bmatrix} + t\begin{bmatrix}-1\\1\\-1\end{bmatrix}$ intersect, and a parametric form of the intersection.

We first find an equation form of the plane given in parametric form. A vector orthogonal to both $\begin{bmatrix}0\\2\\3\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\-1\end{bmatrix}$ is their crossed product.

$$\begin{bmatrix}0\\2\\3\end{bmatrix} \times \begin{bmatrix}-1\\1\\-1\end{bmatrix} = \det\begin{bmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ 0 & 2 & 3\\ -1 & 1 & -1\end{bmatrix} = \begin{bmatrix}-5\\-3\\2\end{bmatrix}$$

So the second plane has an equation form:

$$0 = (\vec{x}-\vec{q})\cdot\vec{v} = \left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix}\right)\cdot\begin{bmatrix}-5\\-3\\2\end{bmatrix} = -5x_1 - 3x_2 + 2x_3 + 6,$$ namely

$$-5x_1 - 3x_2 + 2x_3 = -6.$$

We now solve the system of equations given by the two planes.

$$\begin{cases} x_1 + \frac{3}{5}x_2 + (k-1)x_3 = 1 \\ -5x_1 - 3x_2 + 2x_3 = -6 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{5} & k-1 & 1 \\ -5 & -3 & 2 & -6 \end{array}\right] \xrightarrow{R_2 + 5\cdot R_1} \left[\begin{array}{ccc|c} 1 & \frac{3}{5} & k-1 & 1 \\ 0 & 0 & 5k-3 & -1 \end{array}\right]$$

When $5k-3=0$, the system has no solutions. When $5k-3 \neq 0$ the system will have infinitely many solutions. Thus when $k=\frac{3}{5}$ the planes do not intersect.

When $k \neq \frac{3}{5}$ we can continue simplifying the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{5} & k-1 & 1 \\ 0 & 0 & 5k-3 & -1 \end{array}\right] \xrightarrow{\frac{1}{5k-3}\cdot R_2} \left[\begin{array}{ccc|c} 1 & \frac{3}{5} & k-1 & 1 \\ 0 & 0 & 1 & \frac{-1}{5k-3} \end{array}\right]$$

This row echelon form has exactly one free parameter, so the planes intersect at a line. We can find the line explicitly by writing the system

$$\begin{cases} x_1 + \frac{3}{5}x_2 + (k-1)x_3 = 1 \\ x_3 = \frac{-1}{5k-3} \end{cases}$$

which after substituting the second equation into the first becomes

$$\begin{cases} x_1 + \frac{3}{5}x_2 = \frac{4k-2}{5k-3} \\ x_3 = \frac{-1}{5k-3}. \end{cases}$$

To write a parametric form corresponding to it, we set $x_2 = s$ a parameter.

Then we have:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4k-2}{5k-3} - \frac{3}{5}s \\ s \\ \frac{-1}{5k-3} \end{bmatrix} = \begin{bmatrix} \frac{4k-2}{5k-3} \\ 0 \\ \frac{-1}{5k-3} \end{bmatrix} + s \begin{bmatrix} -\frac{3}{5} \\ 1 \\ 0 \end{bmatrix}.$$

<u>Example:</u> Problems of the form "find the next number in the sequence" make <u>no</u> sense. We can cook up polynomials that produce any number that we want.

A sequence of numbers begins as: 1, 2, 3, 4. Find a polynomial satisfying $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, $f(4) = 4$, and $f(5) = k$ where k is a fixed scalar.

We first need to determine the degree of the polynomial $f(x)$. Since we have five equalities that need to be satisfied, we need a polynomial with five parameters (which will be the unknowns we will be solving for). We need $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, a polynomial of degree four. Then:

$$\begin{cases} 1 = f(1) = a + b + c + d + e \\ 2 = f(2) = 16a + 8b + 4c + 2d + e \\ 3 = f(3) = 81a + 27b + 9c + 3d + e \\ 4 = f(4) = 256a + 64b + 16c + 4d + e \\ k = f(5) = 625a + 125b + 25c + 5d + e \end{cases}$$

is the system of equations we are solving, using Gaussian elimination.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 & 2 \\ 81 & 27 & 9 & 3 & 1 & 3 \\ 256 & 64 & 16 & 4 & 1 & 4 \\ 625 & 125 & 25 & 5 & 1 & k \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{24}\cdot k - \frac{5}{24} \\ 0 & 1 & 0 & 0 & 0 & -\frac{5}{12}\cdot k + \frac{25}{12} \\ 0 & 0 & 1 & 0 & 0 & \frac{35}{24}\cdot k - \frac{175}{24} \\ 0 & 0 & 0 & 1 & 0 & -\frac{25}{12}\cdot k + \frac{137}{12} \\ 0 & 0 & 0 & 0 & 1 & k-5 \end{array}\right]$$

The polynomial we want is then:

$$f(x) = \left(\frac{1}{24}\cdot k - \frac{5}{24}\right)x^4 + \left(-\frac{5}{12}\cdot k + \frac{25}{12}\right)x^3 + \left(\frac{35}{24}\cdot k - \frac{175}{24}\right)x^2 + \left(-\frac{25}{12}\cdot k + \frac{137}{12}\right)x + k - 5.$$

<u>Example:</u> Find the intersection between the line $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$ and the plane

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}.$$

We are solving the following equation:

$$\begin{bmatrix} 1+r \\ r \\ 1+2r \\ 2r \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} s+t \\ 1-s+t \\ 1+s+3t \\ 2t \end{bmatrix}$$

namely the system of equations:

$$\begin{cases} r-s-t=-1 \\ r+s-t=1 \\ 2r-s-3t=0 \\ 2r-2t=0. \end{cases}$$

Its augmented matrix and reduced row echelon form follow.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & -3 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $r=-1$, $s=1$, $t=-1$ give the intersection point. We can check that it is the same for the line and the plane: the intersection point is $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ -2 \end{bmatrix}$

and $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ -2 \end{bmatrix}$.

<u>Example</u>: Find a plane perpendicular to $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$ going through $\begin{bmatrix} 0 \\ -1 \\ -1 \\ 2 \end{bmatrix}$.

(i) In equation form.

(ii) In parametric form.

(i) We are given two directions to which we must be perpendicular, the vectors

$\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$, as well as a point $\vec{q} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 2 \end{bmatrix}$ that we have to go through. An equation form is then:

$$\begin{cases} 0 = (\vec{x}-\vec{q})\cdot\vec{v} = \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ -1 \\ 2 \end{bmatrix}\right)\cdot\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = x_1 - x_2 - 1 + x_3 + 1 \\ 0 = (\vec{x}-\vec{q})\cdot\vec{w} = \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ -1 \\ 2 \end{bmatrix}\right)\cdot\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} = x_1 + x_2 + 1 + 3x_3 + 3 + 2x_4 - 4 \end{cases}$$

or equivalently:

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 + 3x_3 + 2x_4 = 0. \end{cases}$$

(ii) We solve the above system of equations to find a parametric form.

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 2 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Leaving $x_4 = t$ and $x_3 = s$ as parameters, the last augmented matrix gives:

$$\begin{cases} x_1 + 2s + t = 0 \\ x_2 + s + t = 0 \end{cases} \quad \text{equivalently} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - t \\ -s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

<u>Example:</u> Determine if any of the following vectors are parallel to the plane

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

(i) $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

(ii) $\begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix}$. Justify it without solving a system of linear equations.

(iii) $\begin{bmatrix} -1 \\ 1 \\ 5 \\ 1 \end{bmatrix}$. Justify it using projections.

(i) The vector $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ is parallel to the plane if and only if it is a linear combination of $\begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$. We are solving the equation $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix} = s \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix} + t \cdot \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

or equivalently the system of equations:

$$\begin{cases} -2s-t=2 \\ -s-t=-1 \\ s+t=1 \\ 2s+t=1 \end{cases}$$ with augmented matrix $$\left[\begin{array}{cc|c} -2 & -1 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

There are <u>no</u> solutions, so $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ is <u>not</u> a linear combination of $\begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ is <u>not</u> parallel to the plane.

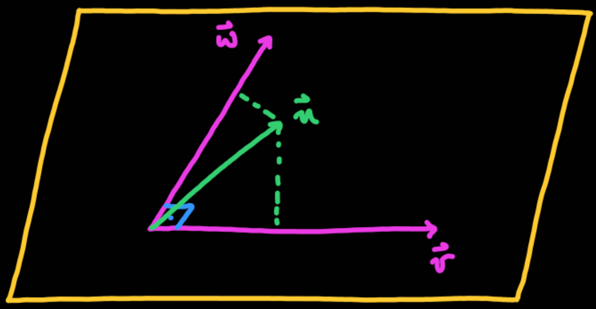
(ii) We can proceed as before, but we are <u>not</u> allowed to solve a system of equations. Sometimes we can also use angles to determine linear dependence or linear independence, and essentially we are just being asked if the vectors $\begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix}$ are linearly dependent or linearly independent. Note that the last vector is orthogonal to the other two because:

$$\begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix} = (-2)\cdot 3 + (-1)\cdot(-1) + 1\cdot 1 + 2\cdot 1 = 0$$

$$\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix} = (-1)\cdot 3 + (-1)\cdot(-3) + (-1)\cdot 1 + 1\cdot 1 = 0$$

so $\begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix}$ is <u>not</u> a linear combination of $\begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix}$ is <u>not</u> parallel to the plane.

(iii) The vector $\begin{bmatrix}-1\\1\\5\\1\end{bmatrix}$ is parallel to the plane if and only if it is equal to the sum of its projection onto two perpendicular vectors inside the plane.



We choose $\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}$ to be the first direction. We then decompose $\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}$ into a component parallel to $\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}$ and a component orthogonal to $\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}$. The parallel component is:

$$\text{proj}_{\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}}\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix} = \left(\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}\right)\cdot\frac{1}{\left\|\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}\right\|^2}\cdot\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix} = \frac{2+1-1+2}{1+1+1+1}\cdot\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}.$$

The orthogonal component is:

$$\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix} - \text{proj}_{\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}}\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix} = \begin{bmatrix}-2\\-1\\1\\2\end{bmatrix} - \begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\2\\1\end{bmatrix}.$$

We now compute the projections:

$$\text{proj}_{\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}}\begin{bmatrix}-1\\1\\5\\1\end{bmatrix} = \left(\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\5\\1\end{bmatrix}\right)\cdot\frac{1}{\left\|\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}\right\|^2}\cdot\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix} = \frac{1-1-5+1}{1+1+1+1}\cdot\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix} = -\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}$$

$$\operatorname{proj}_{\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}}\begin{bmatrix}-1\\1\\5\\1\end{bmatrix}=\left(\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\5\\1\end{bmatrix}\right)\cdot\frac{1}{\left\|\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}\right\|^2}\cdot\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}=\frac{1+0+10+1}{1+0+4+1}\cdot\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}=2\cdot\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}$$

and we add them up:

$$\operatorname{proj}_{\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}}\begin{bmatrix}-1\\1\\5\\1\end{bmatrix}+\operatorname{proj}_{\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}}\begin{bmatrix}-1\\1\\5\\1\end{bmatrix}=-\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}+2\cdot\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}=\begin{bmatrix}-1\\1\\5\\1\end{bmatrix}.$$

Since both projections are in the plane determined by $\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}$ and $\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}$, and $\begin{bmatrix}-1\\1\\5\\1\end{bmatrix}$ is a linear combination of those projections, the vector $\begin{bmatrix}-1\\1\\5\\1\end{bmatrix}$ is parallel to the plane.

Example: Find the smallest distance between the lines $\vec{x}=\begin{bmatrix}1\\1\end{bmatrix}+s\begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{x}=\begin{bmatrix}-1\\2\end{bmatrix}+t\begin{bmatrix}2\\2\end{bmatrix}$.

These lines do <u>not</u> intersect because $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$ are parallel. The vector $\vec{v}=\begin{bmatrix}-1\\1\end{bmatrix}$ is perpendicular to both of them, we can find it by solving the equation $0=\begin{bmatrix}1\\1\end{bmatrix}\cdot\vec{v}$. A point in the first line is $\begin{bmatrix}1\\1\end{bmatrix}$, a point in the second line is $\begin{bmatrix}-1\\2\end{bmatrix}$, so the vector $\vec{w}=\begin{bmatrix}-1\\2\end{bmatrix}-\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}$ points from the first line to the second line. The length of the projection:

$$\operatorname{proj}_{\vec{v}}\vec{w}=(\vec{v}\cdot\vec{w})\frac{\vec{v}}{\|\vec{v}\|^2}=\frac{2+1}{1+1}\cdot\begin{bmatrix}-1\\1\end{bmatrix}=\frac{3}{2}\begin{bmatrix}-1\\1\end{bmatrix},$$

namely:

$$\left\| \frac{3}{2} \begin{bmatrix} -1 \\ \\ 1 \end{bmatrix} \right\| = \sqrt{\frac{9}{4} \cdot 1 + \frac{9}{4} \cdot 1} = \frac{3}{2} \cdot \sqrt{2},$$

is the distance between the lines.

<u>Example:</u> Find the smallest distance between the lines $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and

$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$

As before, the idea is to find a vector perpendicular to both lines, find a vector from one line to the other, and compute the length of the projection. The cross product:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

is perpendicular to both lines, the vector $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ points from the first line to the second, and the length of the projection:

$$\operatorname{proj}_{\vec{v}} \vec{w} = (\vec{v} \cdot \vec{w}) \frac{\vec{v}}{\|\vec{v}\|^2} = \frac{-2+0-1}{1+0+1} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{-3}{2} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

namely:

$$\left\| \frac{-3}{2} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\| = \frac{3}{2} \cdot \sqrt{1+0+1} = \frac{3\sqrt{2}}{2},$$ is the distance between the lines.

Example: Find the smallest distance between the lines $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$.

The idea is still the same, find a vector perpendicular to both lines, find a vector from one line to the other, and compute the length of the projection. However, since now there is a two dimensional plane parallel to each line, we need to be more careful when finding a vector perpendicular to both of them. For this, it is useful to <u>first</u> find the general vector pointing from the first line to the second line, and <u>then</u> impose that this vector be perpendicular to both lines. The general vector pointing from the first line to the second is:

$$\vec{\omega} = \left(\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right) - \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

Imposing perpendicularity to the lines gives:

$$\begin{cases} 0 = \vec{\omega} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = -2+1+0-1+2t-t+t-2t-s-s-s-s = -2-4s \\ 0 = \vec{\omega} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} = -4-1+0+2+4t+t+t+4t-2s+s-s+2s = -3+10t \end{cases}$$

whose solution is $s = \frac{-1}{2}$ and $t = \frac{3}{10}$. Substituting these into $\vec{w}$ gives:

$$\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \frac{3}{10} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -9/10 \\ -6/5 \\ 4/5 \\ 11/10 \end{bmatrix}$$

wose length:

$$\left\| \begin{bmatrix} -9/10 \\ -6/5 \\ 4/5 \\ 11/10 \end{bmatrix} \right\| = \frac{1}{10} \cdot \sqrt{81 + 144 + 64 + 121} = \frac{\sqrt{410}}{10} = \sqrt{\frac{41}{10}}$$

is the distance between the two lines.

<u>Example:</u> Find the smallest natural numbers x_1, x_2, x_3, x_4 that balance the following chemical reaction: $x_1 C_6H_{12}O_6 + x_2 O_2 \rightarrow x_3 CO_2 + x_4 H_2O$.

We want to have the same number of each element before and after the reaction.

$$\begin{array}{ll} C: & 6x_1 = x_3 \\ H: & 12x_1 = 2x_4 \\ O: & 6x_1 + 2x_2 = 2x_3 + x_4 \end{array} \qquad \text{namely} \qquad \begin{cases} 6x_1 - x_3 = 0 \\ 12x_1 - 2x_4 = 0 \\ 6x_1 + 2x_2 - 2x_3 - x_4 = 0 \end{cases}$$

The augmented matrix and its ref are the following.

$$\left[\begin{array}{cccc|c} 6 & 0 & -3 & 0 & 0 \\ 12 & 0 & 0 & -2 & 0 \\ 6 & 2 & -2 & -1 & 0 \end{array}\right] \xrightarrow{\text{ref}} \left[\begin{array}{cccc|c} 6 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Then:

$$\begin{cases} 6x_1 - x_4 = 0 \\ 2x_2 - x_3 - x_4 = 0 \\ x_3 - x_4 = 0 \end{cases} \quad \text{so setting } x_4 \text{ as a parameter gives} \quad \begin{cases} x_1 = \frac{1}{6} x_4 \\ x_2 = x_4 \\ x_3 = x_4 \\ x_4 = x_4. \end{cases}$$

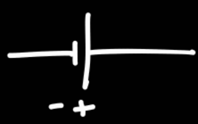
The smallest natural solutions occur for $x_4 = 6$. Then:

$$C_6H_{12}O_6 + 6\,O_2 \rightarrow 6\,CO_2 + 6\,H_2O.$$

3.5. Resistor networks

We can use linear algebra and systems of equations to determine the current and voltage drop anywhere in a resistor network. A resistor network is an electric circuit that only involves resistances, batteries, and sources of current. These look as follows:

Resistance:  denoted R, measured in Ohms Ω.

Battery:  denoted V, measured in Volts V.

Current source:  denoted I, measured in Ampere A.

They follow Ohm's Law: $V = IR$ or the voltage drop is proportional to the resistance times the current. Problems involving resistor networks will always involve solving the current flowing through a circuit. We will always use two physical laws:

1) Conservation of current in a circuit with resistances in sequence.

2) Conservation of voltage drop in a circuit with resistors in parallel.

These will give us a system of equations, where the unknowns will be currents going through cables and voltage drop along a current source. The procedure to set up the system of equations is:

(1) Identify the inner loops of the system.

(2) Follow a closed path along each loop of the circuit, without repeating edges.

(3) Label first the edge having the battery or current source.

(4) Every time we encounter a node, relabel the edges, so that every edge has a distinct name associated to it.

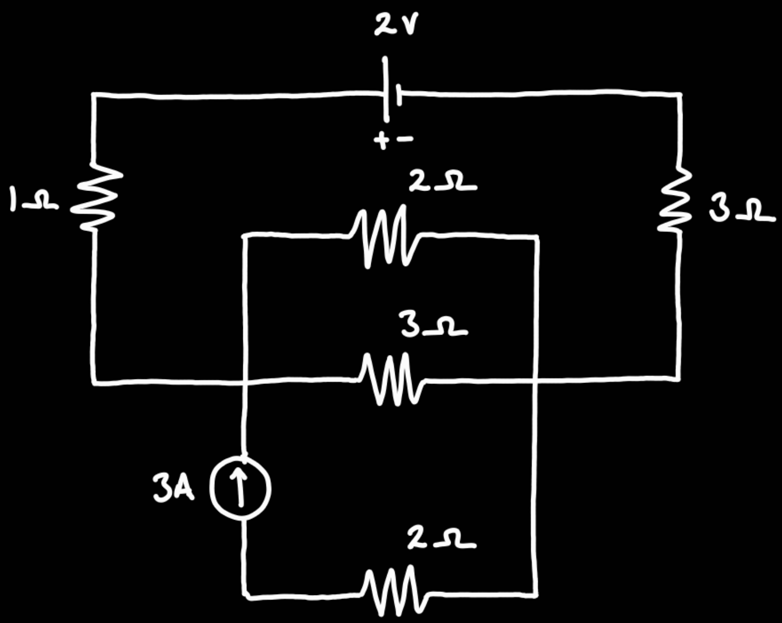
(5) With all edges labeled, follow again the same closed paths along each loop. When a resistance is encountered, use Ohm's Law to find the voltage. When a source of current is encountered, label its unknown voltage drop.

(6) In each loop, add the voltages of the labels in the direction we followed,

subtract the voltages of the labels in the opposite direction, and equal zero.

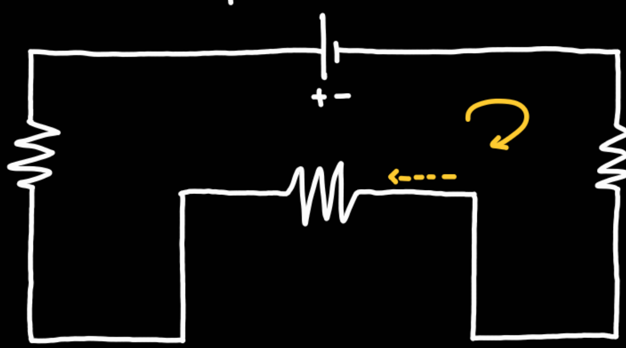
(7) For each node, add the incoming labels, subtract the outgoing labels, and equal zero.

<u>Example:</u> Solve the following resistor network.

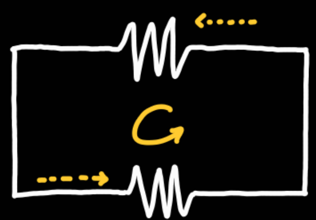


There are three inner loops.

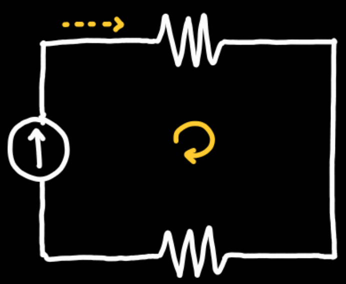
(L1)



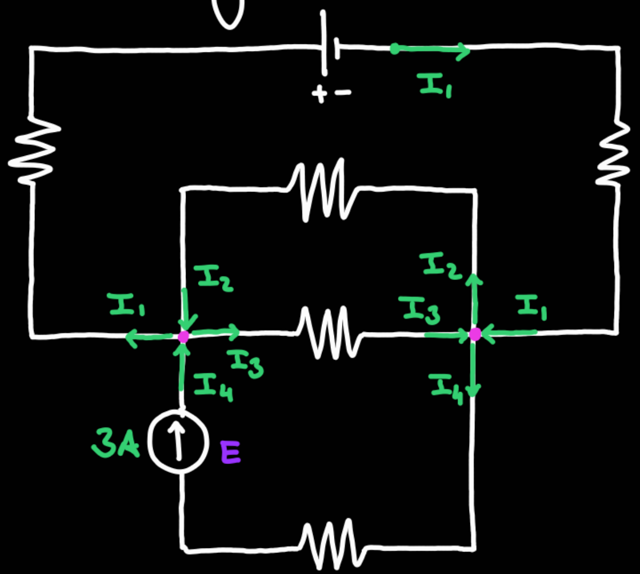
(L2)



(L3)



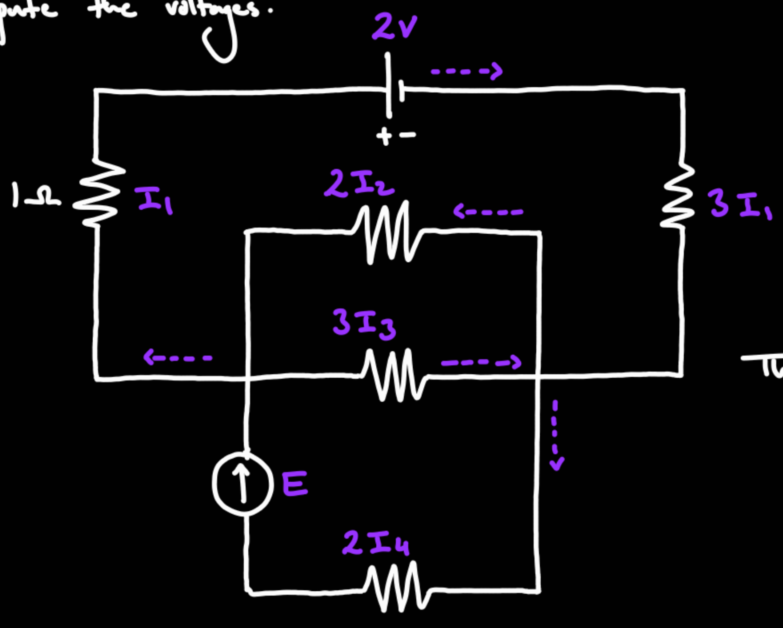
We now label the edges and the current sources.



These give:

$$\begin{cases} I_1 - I_2 + I_3 - I_4 = 0 \\ I_4 = 3 \\ -I_1 + I_2 - I_3 + I_4 = 0 \end{cases}$$

We compute the voltages.



These give:

$$\begin{cases} 3I_1 + 2I_2 + I_1 + 2 = 0 \\ 2I_2 + 3I_3 = 0 \\ 2I_4 + E + 3I_3 = 0 \end{cases}$$

Putting together all the equations we get:

$$\begin{cases} I_1 - I_2 + I_3 - I_4 = 0 \\ I_4 = 3 \\ -I_1 + I_2 - I_3 + I_4 = 0 \\ 3I_1 + 2I_2 + I_1 + 2 = 0 \\ 2I_2 + 3I_3 = 0 \\ 2I_4 + E + 3I_3 = 0 \end{cases}$$

namely $I_4 = 3$ and

$$\begin{cases} I_1 - I_2 + I_3 = 3 \\ 4I_1 + 2I_2 = -2 \\ 2I_2 + 3I_3 = 0 \\ 3I_3 + E = -6. \end{cases}$$

The augmented matrix and its reduced row echelon form are:

$$\begin{array}{c}
\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 3 \\ 4 & 2 & 0 & 0 & -2 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 & -6 \end{array}\right] \\
\begin{array}{cccc} I_1 & I_2 & I_3 & E \end{array}
\end{array}
\xrightarrow{\text{rref}}
\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{4}{13} \\ 0 & 1 & 0 & 0 & \frac{-21}{13} \\ 0 & 0 & 1 & 0 & \frac{14}{13} \\ 0 & 0 & 0 & 1 & \frac{-120}{13} \end{array}\right]$$

which gives:

$$I_1 = \frac{4}{13}, \quad I_2 = \frac{-21}{13}, \quad I_3 = \frac{14}{13}, \quad E = \frac{-120}{13}.$$

The negative signs mean that the orientation we chose for that specific current is wrong, and that the current goes in fact the other way. The negative sign in the resistance of the source of current tells us that the orientation of the currents we chose encounter first the negative pole of a voltage drop, and then the positive pole. Altogether, we have the following circuit.