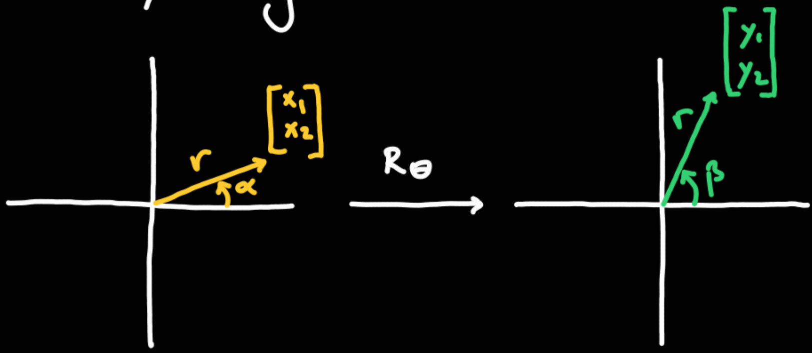
Arguably the three most natural operations we can do geometrically are rotations, projections, and reflections. We now interpret them as linear transformations.

<u>Rotations:</u> Denote by $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ the functions that inputs a vector and outputs its rotation by θ degrees counter-clockwise. Geometrically, we have the following:



$; \quad R_\theta\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$

where $\beta = \alpha + \theta$. Also:

$$x_1 = r \cdot \cos\theta, \quad x_2 = r \cdot \sin\theta, \quad y_1 = r \cdot \cos\beta, \quad y_2 = r \cdot \sin\beta.$$

Then:

$$y_1 = r \cdot \cos\beta = r \cdot \cos(\alpha + \theta) = r \cdot \cos\alpha \cdot \cos\theta - r \cdot \sin\alpha \cdot \sin\theta = x_1 \cdot \cos\theta - x_2 \cdot \sin\theta$$

trigonometric addition formulas

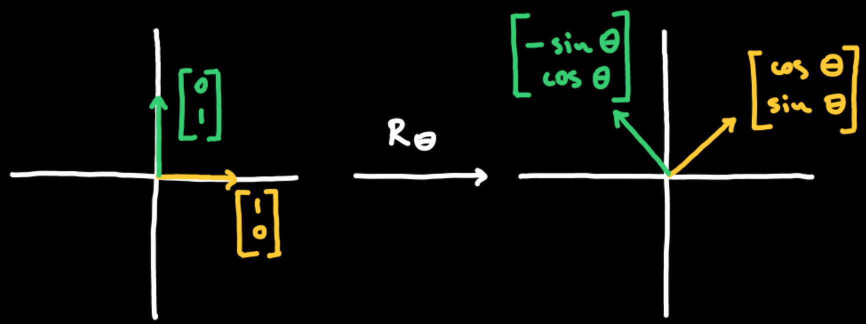
$$y_2 = r \cdot \sin\beta = r \cdot \sin(\alpha + \theta) = r \cdot \cos\alpha \cdot \sin\theta + r \cdot \sin\alpha \cdot \cos\theta = x_1 \cdot \sin\theta + x_2 \cdot \cos\theta$$

so we can write:

$$R_\theta\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \cdot \cos\theta - x_2 \cdot \sin\theta \\ x_1 \cdot \sin\theta + x_2 \cdot \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

whence rotation is a linear transformation, and that is the associated matrix.

If we already knew that rotation was a linear transformation, we could have obtained the associated matrix by seeing the output of the standard basis:



so the matrix associated to R_θ would be:

$$\left[R_\theta\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \; R_\theta\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

<u>Easy challenge:</u> Find the matrices associated to yaw, pitch, and roll, namely:

1) Rotation by θ counterclockwise from the positive x axis.

2) Rotation by θ counterclockwise from the positive y axis.

3) Rotation by θ counterclockwise from the positive z axis.

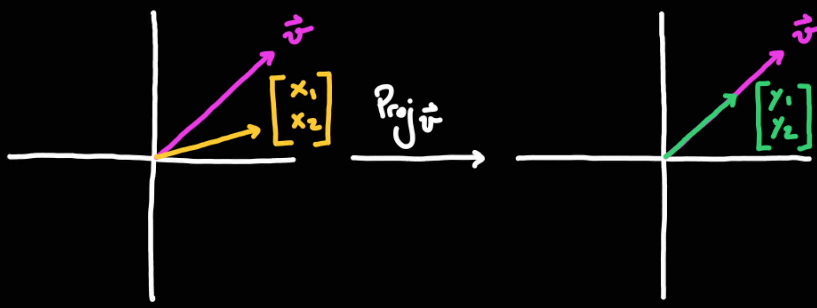
<u>Hard challenge:</u> Find the matrix associated to the rotation by θ from the negative $\vec{v}$ axis, where $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is any vector in $\mathbb{R}^3$.

<u>Spoilers:</u>

Easy: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$, $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$, $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Hard: $\dfrac{1}{v_1^2+v_2^2+v_3^2}\begin{bmatrix} v_1^2(1-\cos\theta)+\cos\theta & v_2v_1(1-\cos\theta)-v_3\sin\theta & v_3v_1(1-\cos\theta)+v_3\sin\theta \\ v_1v_2(1-\cos\theta)+v_3\sin\theta & v_2^2(1-\cos\theta)+\cos\theta & v_3v_2(1-\cos\theta)-v_1\sin\theta \\ v_1v_3(1-\cos\theta)-v_2\sin\theta & v_2v_3(1-\cos\theta)+v_1\sin\theta & v_3^2(1-\cos\theta)+\cos\theta \end{bmatrix}.$

<u>Projections:</u> Denote by $\mathrm{Proj}_{\vec{v}}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ the function that inputs a vector and outputs its projection onto $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Geometrically, we have the following:



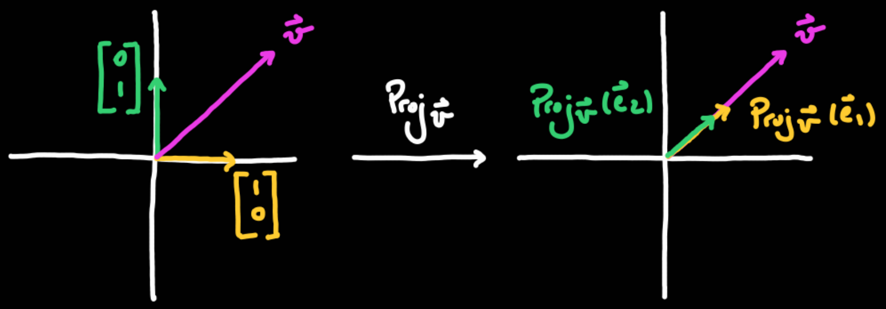
and algebraically we have:

$$\mathrm{Proj}_{\vec{v}}(\vec{x}) = \frac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{x_1v_1+x_2v_2}{v_1^2+v_2^2}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{v_1^2+v_2^2}\begin{bmatrix} v_1^2x_1+v_1v_2x_2 \\ v_1v_2x_1+v_2^2x_2 \end{bmatrix} =$$

$$= \frac{1}{v_1^2+v_2^2}\begin{bmatrix} v_1^2 & v_1v_2 \\ v_2v_1 & v_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{v_1^2}{v_1^2+v_2^2} & \frac{v_1v_2}{v_1^2+v_2^2} \\ \frac{v_2v_1}{v_1^2+v_2^2} & \frac{v_2^2}{v_1^2+v_2^2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

whence projection is a linear transformation, and that is the associated matrix.

If we already knew that projection was a linear transformation, we could have obtained the associated matrix by seeing the output of the standard basis:



because:

$$\mathrm{Proj}_{\vec{v}}(\vec{e}_1) = \frac{(\vec{e}_1 \cdot \vec{v})}{\|\vec{v}\|^2} \cdot \vec{v} = \frac{v_1}{v_1^2 + v_2^2}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1^2/(v_1^2+v_2^2) \\ v_1 v_2/(v_1^2+v_2^2) \end{bmatrix}$$

$$\mathrm{Proj}_{\vec{v}}(\vec{e}_2) = \frac{(\vec{e}_2 \cdot \vec{v})}{\|\vec{v}\|^2} \cdot \vec{v} = \frac{v_2}{v_1^2 + v_2^2}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 v_2/(v_1^2+v_2^2) \\ v_2^2\ /(v_1^2+v_2^2) \end{bmatrix}$$

so the matrix associated to $\mathrm{Proj}_{\vec{v}}$ would be:

$$\begin{bmatrix} \mathrm{Proj}_{\vec{v}}(\vec{e}_1) & \mathrm{Proj}_{\vec{v}}(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} \frac{v_1^2}{v_1^2+v_2^2} & \frac{v_1 v_2}{v_1^2+v_2^2} \\ \frac{v_1 v_2}{v_1^2+v_2^2} & \frac{v_2^2}{v_1^2+v_2^2} \end{bmatrix}.$$

<u>Easy challenge:</u> Find the matrix associated to the projection onto $\vec{v}$, where

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is any vector in $\mathbb{R}^3$.

<u>Medium challenge:</u> Find the matrix associated to the projection onto the plane

containing $\vec{v}$ and $\vec{w}$, where $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ are any non-colinear vectors

in $\mathbb{R}^3$.

<u>Spoilers:</u>

Easy: $\dfrac{1}{v_1^2+v_2^2+v_3^2}\begin{bmatrix} v_1^2 & v_2v_1 & v_3v_1 \\ v_1v_2 & v_2^2 & v_3v_2 \\ v_1v_3 & v_2v_3 & v_3^2 \end{bmatrix}$

Medium: $\dfrac{1}{L}\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ where $L=(v_2w_3-v_3w_2)^2+(v_3w_1-v_1w_3)^2+(v_1w_2-v_2w_1)^2$,

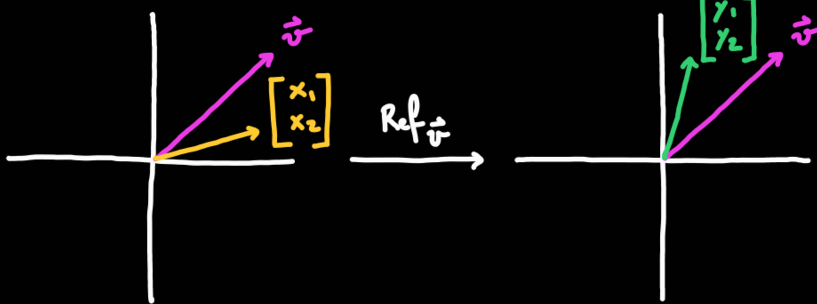
$a=(v_3w_1-v_1w_3)^2+(v_1w_2-v_2w_1)^2$,

$b=(w_3v_2-w_2v_3)(v_3w_1-v_1w_3)$, $c=(w_3v_2-w_2v_3)(v_1w_2-v_2w_1)$,

$d=(v_2w_3-v_3w_2)^2+(v_1w_2-v_2w_1)^2$, $e=(w_1v_3-w_3v_1)(v_1w_2-v_2w_1)$,

$f=(v_2w_3-v_3w_2)^2+(v_3w_1-v_1w_3)^2$.

<u>Reflections:</u> Denote by $\mathrm{Ref}_{\vec v}:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ the function that inputs a vector and outputs its reflection across the line with direction $\vec v=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Geometrically, we have the following:



and algebraically we have:

$$\mathrm{Ref}_{\vec v}(\vec x)=\mathrm{Proj}_{\vec v}(\vec x)-\left(\vec x-\mathrm{Proj}_{\vec v}(\vec x)\right)=2\,\mathrm{Proj}_{\vec v}(\vec x)-\vec x=$$

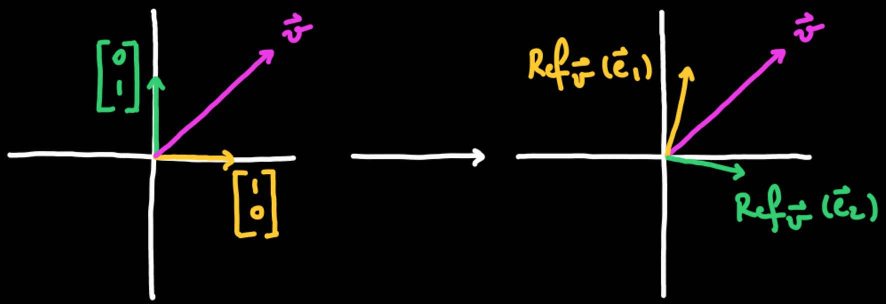
$$=2\begin{bmatrix} \dfrac{v_1^2}{v_1^2+v_2^2} & \dfrac{v_1v_2}{v_1^2+v_2^2} \\ \dfrac{v_2v_1}{v_1^2+v_2^2} & \dfrac{v_2^2}{v_1^2+v_2^2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=$$

$$= \left(\begin{bmatrix} \frac{2v_1^2}{v_1^2+v_2^2} & \frac{2v_1v_2}{v_1^2+v_2^2} \\ \frac{2v_2v_1}{v_1^2+v_2^2} & \frac{2v_2^2}{v_1^2+v_2^2} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{v_1^2-v_2^2}{v_1^2+v_2^2} & \frac{2v_1v_2}{v_1^2+v_2^2} \\ \frac{2v_2v_1}{v_1^2+v_2^2} & \frac{v_2^2-v_1^2}{v_1^2+v_2^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

whence reflection is a linear transformation, and that is the associated matrix.

If we already knew that reflection was a linear transformation, we could have obtained the associated matrix by seeing the output of the standard basis:



because:

$$Ref_{\vec{v}}(\vec{e}_1) = 2\,Proj_{\vec{v}}(\vec{e}_1) - \vec{e}_1 = \begin{bmatrix} 2v_1^2/(v_1^2+v_2^2) \\ 2v_1v_2/(v_1^2+v_2^2) \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (v_1^2-v_2^2)/(v_1^2+v_2^2) \\ 2v_1v_2/(v_1^2+v_2^2) \end{bmatrix}$$

$$Ref_{\vec{v}}(\vec{e}_2) = 2\,Proj_{\vec{v}}(\vec{e}_2) - \vec{e}_2 = \begin{bmatrix} 2v_1v_2/(v_1^2+v_2^2) \\ 2v_2^2/(v_1^2+v_2^2) \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2v_1v_2/(v_1^2+v_2^2) \\ (v_2^2-v_1^2)/(v_1^2+v_2^2) \end{bmatrix}$$

so the matrix associated to $Ref_{\vec{v}}$ would be:

$$\begin{bmatrix} Ref_{\vec{v}}(\vec{e}_1) & Ref_{\vec{v}}(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} \frac{v_1^2-v_2^2}{v_1^2+v_2^2} & \frac{2v_1v_2}{v_1^2+v_2^2} \\ \frac{2v_1v_2}{v_1^2+v_2^2} & \frac{v_2^2-v_1^2}{v_1^2+v_2^2} \end{bmatrix}.$$

__Easy challenge:__ Find the matrix associated to the reflection across the line with direction $\vec{v}$, where $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is any vector in $\mathbb{R}^3$.

__Medium challenge:__ Find the matrix associated to the reflection across the plane containing $\vec{v}$ and $\vec{w}$, where $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ are any non-colinear vectors in $\mathbb{R}^3$.

__Spoilers:__

Easy: $$\frac{1}{v_1^2+v_2^2+v_3^2}\begin{bmatrix} v_1^2-v_2^2-v_3^2 & 2v_2v_1 & 2v_3v_1 \\ 2v_1v_2 & v_2^2-v_1^2-v_3^2 & 2v_3v_2 \\ 2v_1v_3 & 2v_2v_3 & v_3^2-v_1^2-v_2^2 \end{bmatrix}$$

Medium: $$\frac{1}{L}\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$ where $L = (v_2w_3-v_3w_2)^2+(v_3w_1-v_1w_3)^2+(v_1w_2-v_2w_1)^2$,

$a = -(v_2w_3-v_3w_2)^2+(v_3w_1-v_1w_3)^2+(v_1w_2-v_2w_1)^2$,

$b = 2(v_2w_3-v_3w_2)(v_1w_3-v_3w_1)$, $c = 2(v_2w_3-v_3w_2)(v_2w_1-v_1w_2)$,

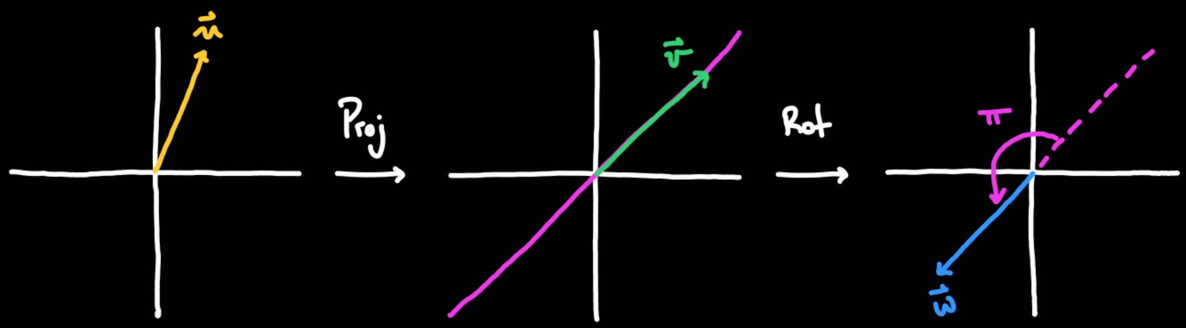
$d = (v_2w_3-v_3w_2)^2-(v_3w_1-v_1w_3)^2+(v_1w_2-v_2w_1)^2$, $e = 2(v_3w_1-v_1w_3)(v_2w_1-v_1w_2)$,

$f = (v_2w_3-v_3w_2)^2+(v_3w_1-v_1w_3)^2-(v_1w_2-v_2w_1)^2$.

Now that we have a multitude of linear transformations, we will see what happens when we compose them, namely when we do one after another.

Example: Let $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the function $f(\vec{x}) = \mathrm{Rot}_\pi\left(\mathrm{Proj}_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}(\vec{x})\right)$, namely first projecting onto the line with direction $\begin{bmatrix}1\\1\end{bmatrix}$ and then rotating by π.

Geometrically:



and algebraically:

$$f(\vec{x}) = \mathrm{Rot}_\pi\left(\mathrm{Proj}_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}(\vec{x})\right) = \mathrm{Rot}_\pi\left(\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \mathrm{Rot}_\pi\begin{bmatrix}\frac{x_1+x_2}{2}\\ \frac{x_1+x_2}{2}\end{bmatrix} =$$

$$= \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{x_1+x_2}{2}\\ \frac{x_1+x_2}{2}\end{bmatrix} = \begin{bmatrix}\frac{-x_1-x_2}{2}\\ \frac{-x_1-x_2}{2}\end{bmatrix} = \begin{bmatrix}\frac{-1}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{-1}{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} =$$

$$= \Bigg(\underbrace{\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}}_{\mathrm{Rot}_\pi}\underbrace{\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}}_{\mathrm{Proj}_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}}\Bigg)\begin{bmatrix}x_1\\x_2\end{bmatrix}.$$

The first interesting thing we see is that f is a linear transformation. The second interesting thing we see is that its associated matrix is the matrix multiplication of the matrices associated to Rot_π and $\mathrm{Proj}_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}$, in that order. Both of these observations are true in general, as we now see.

Given $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ linear transformations, the composition $gf : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is a linear transformation. For all r in $\mathbb{R}$ and $\vec{x}, \vec{y}$ in $\mathbb{R}^n$:

(i) $(gf)(r\vec{x}) = g(f(r\vec{x})) \underset{\uparrow\, f \text{ linear}}{=} g(r f(\vec{x})) \underset{\uparrow\, g \text{ linear}}{=} r g(f(\vec{x})) = r(gf)(\vec{x})$

(ii) $(gf)(\vec{x}+\vec{y}) = g(f(\vec{x}+\vec{y})) \overset{f \text{ linear}\,\downarrow}{=} g(f(\vec{x})+f(\vec{y})) \overset{g \text{ linear}\,\downarrow}{=} g(f(\vec{x})) + g(f(\vec{y})) = (gf)(\vec{x}) + (gf)(\vec{y})$.

Let A be an $m \times n$ matrix such that $f(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$ and let B be a $p \times m$ matrix such that $g(\vec{x}) = B\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^m$. Then $(gf)(\vec{x}) = (BA)\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^m$. The formal reason requires using the rigorous definition of matrix multiplication and is outside the scope of the course. We now illustrate this with an example.

<u>Example:</u> Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation given by $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation given by $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Their composition $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given as follows:

$$\begin{array}{cccccc} h: & \mathbb{R}^2 & \xrightarrow{\ f\ } & \mathbb{R}^2 & \xrightarrow{\ g\ } & \mathbb{R}^2 \\ & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \longmapsto & \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} & \longmapsto & \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{array}$$

where:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

We can now see h from two perspectives.

(i) h is doing first f and then g, namely:

$$h\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = g\left(f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\right) = g\left(\begin{bmatrix} x_1+2x_2 \\ 3x_1+4x_2 \end{bmatrix}\right) = \begin{bmatrix} 23x_1+34x_2 \\ 31x_1+46x_2 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(ii) the output $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = h\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$ is given by solving the systems of equations

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$ namely:

$$\begin{cases} y_1 = x_1+2x_2 \\ y_2 = 3x_1+4x_2 \end{cases} \text{ and } \begin{cases} z_1 = 5y_1+6y_2 \\ z_2 = 7y_1+8y_2 \end{cases},$$ which by substitution give

$$\begin{cases} z_1 = 5(x_1+2x_2)+6(3x_1+4x_2) = 23x_1+34x_2 \\ z_2 = 7(x_1+2x_2)+8(3x_1+4x_2) = 31x_1+46x_2 \end{cases}.$$

Matrix multiplication is defined to capture precisely this substitution, and indeed

the system $\begin{cases} z_1 = 23x_1+34x_2 \\ z_2 = 31x_1+46x_2 \end{cases}$ corresponds to the equality $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The composition $h = gf$ has associated matrix $\underbrace{\begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}}_{h} = \underbrace{\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}}_{g}\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{f}$.

<u>Example:</u> Give the matrices of $\mathrm{Ref}_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}\mathrm{Ref}_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\mathrm{Rot}_\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

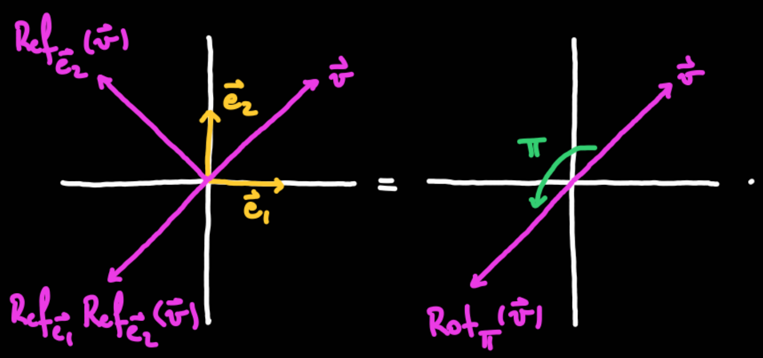
Since the matrices associated to $\mathrm{Ref}_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\mathrm{Ref}_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, the matrix of $\mathrm{Ref}_{\begin{bmatrix}1\\0\end{bmatrix}}\mathrm{Ref}_{\begin{bmatrix}0\\1\end{bmatrix}}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the product

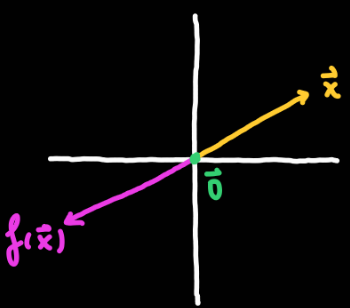
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. The matrix associated to $\mathrm{Rot}_\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is

$\begin{bmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. They are the same, so $\mathrm{Ref}_{\begin{bmatrix}1\\0\end{bmatrix}}\mathrm{Ref}_{\begin{bmatrix}0\\1\end{bmatrix}} = \mathrm{Rot}_\pi$.

Geometrically:



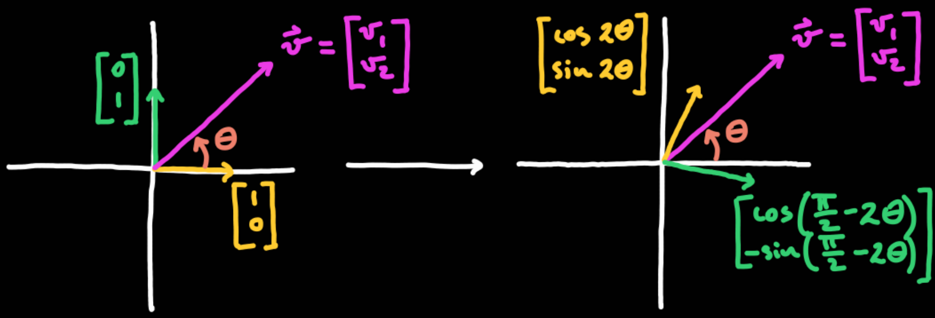
Are there other geometric operations coinciding with these? Yes! For example, scaling by a factor of -1 and reflection across the origin $\vec{0}$. Scaling by -1 is just multiplying by -1, so it is linear, and its associated matrix has $-\vec{e}_1$ in the first column and $-\vec{e}_2$ in the second column, namely $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Reflection across the origin is:



where $f(\vec{x}) = \vec{x} - 2\vec{x} = -\vec{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

In general, a rotation in n-dimensions can be obtained as the composition of

$n+1$ reflections. Let's see this in two dimensions. Since we wrote rotations with angles, let's translate the matrix we obtained for reflections so that they depend on the angle between the vector in the direction of the line and the x-axis (instead of depending on the coordinates of the vector). Geometrically:



and we can simplify $\cos\left(\frac{\pi}{2} - 2\theta\right) = -\sin(-2\theta) = \sin 2\theta$, $-\sin\left(\frac{\pi}{2} - 2\theta\right) = -\cos(-2\theta) = -\cos 2\theta$, to obtain that $Ref_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ the reflection across the line angled θ degrees is:

$$\begin{bmatrix} Ref_\theta(\vec{e}_1) & Ref_\theta(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

This works because we already know that reflections are linear. We could have also used that $v_1 = r \cdot \cos\theta$ and $v_2 = r \cdot \sin\theta$ to simplify:

$$\frac{v_1^2 - v_2^2}{v_1^2 + v_2^2} = \frac{r^2 \cdot \cos^2\theta - r^2 \cdot \sin^2\theta}{r^2 \cdot \cos^2\theta + r^2 \cdot \sin^2\theta} = \frac{r^2}{r^2} \cdot \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta} = \cos 2\theta$$

$$\frac{2 v_1 v_2}{v_1^2 + v_2^2} = \frac{2 r^2 \cos(\theta) \cdot \sin(\theta)}{r^2 \cdot \cos^2\theta + r^2 \cdot \sin^2\theta} = \frac{r^2}{r^2} \cdot \frac{2\cos(\theta) \cdot \sin(\theta)}{\cos^2(\theta) + \sin^2(\theta)} = \sin 2\theta$$

$$\frac{v_2^2 - v_1^2}{v_1^2 + v_2^2} = \frac{r^2 \cdot \sin^2\theta - r^2 \cdot \cos^2\theta}{r^2 \cdot \cos^2\theta + r^2 \cdot \sin^2\theta} = \frac{r^2}{r^2} \cdot \frac{\sin^2\theta - \cos^2\theta}{\cos^2\theta + \sin^2\theta} = -\cos 2\theta$$

double angle formula

obtaining:

$$\begin{bmatrix} \frac{v_1^2 - v_2^2}{v_1^2+v_2^2} & \frac{2v_1v_2}{v_1^2+v_2^2} \\ \frac{2v_1v_2}{v_1^2+v_2^2} & \frac{v_2^2-v_1^2}{v_1^2+v_2^2} \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Then the composition of the reflections across the lines with angles α and β with respect to the x-axis is the following multiplication of matrices:

$$\begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix}\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix} =$$

$$= \begin{bmatrix} \cos(2\beta)\cos(2\alpha) + \sin(2\beta)\sin(2\alpha) & \cos(2\beta)\sin(2\alpha) - \sin(2\beta)\cos(2\alpha) \\ \sin(2\beta)\cos(2\alpha) - \cos(2\beta)\sin(2\alpha) & \sin(2\beta)\sin(2\alpha) + \cos(2\beta)\cos(2\alpha) \end{bmatrix} =$$

angle sum formula

$$= \begin{bmatrix} \cos(2\beta - 2\alpha) & -\sin(2\beta-2\alpha) \\ \sin(2\beta-2\alpha) & \cos(2\beta-2\alpha) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

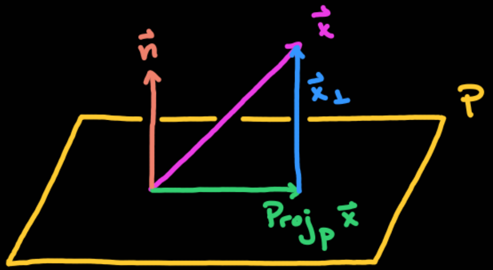
$\theta = 2(\beta - \alpha)$

namely a rotation of $\theta = 2(\beta-\alpha)$ degrees counterclockwise. This means that a rotation of θ counterclockwise equals doing first a reflection across a line that is α degrees from the x-axis and then doing a reflection across a line that is β degrees from the x-axis, as long as $\theta = 2(\beta - \alpha)$.

Let's do some examples in three dimensions.

<u>Example:</u> Find the matrix associated to the projection onto the plane $x+y+z=0$.

Geometrically, we have:



where $\vec{x} = \mathrm{Proj}_P(\vec{x}) + \vec{x}_\perp$, where $\vec{x}_\perp$ is the shortest vector from $\vec{x}$ to P.

In general, to compute the projection $\mathrm{Proj}_F(\vec{x})$ where F is a geometric figure given by a system of equations, we first compute $\vec{x}_\perp$ the shortest vector from $\vec{x}$ to F (namely the one whose length is the distance between $\vec{x}$ and F) and then $\mathrm{Proj}_F(\vec{x}) = \vec{x} - \vec{x}_\perp$.

In three dimensions and for projection onto a plane, we can use the unit vector normal to the plane to make our life easier. The plane is $x+y+z=0$, so the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is normal to the plane, so the unit vector normal to the plane is $\vec{n} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$. Then:

$$\vec{x}_\perp = \mathrm{Proj}_{\vec{n}}(\vec{x}) = \frac{\vec{x}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \frac{\frac{x_1}{\sqrt{3}} + \frac{x_2}{\sqrt{3}} + \frac{x_3}{\sqrt{3}}}{1}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}x_1+x_2+x_3\\x_1+x_2+x_3\\x_1+x_2+x_3\end{bmatrix},\ \text{so}$$

$$\mathrm{Proj}_{x+y+z=0}(\vec{x}) = \vec{x} - \vec{x}_\perp = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} - \frac{1}{3}\begin{bmatrix}x_1+x_2+x_3\\x_1+x_2+x_3\\x_1+x_2+x_3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2x_1-x_2-x_3\\-x_1+2x_2-x_3\\-x_1-x_2-2x_3\end{bmatrix} =$$
$$= \begin{bmatrix}2/3 & -1/3 & -1/3\\-1/3 & 2/3 & -1/3\\-1/3 & -1/3 & 2/3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}.$$

Alternatively, since projections are linear, we could have computed:

$$\mathrm{Proj}_{x+y+z=0}(\vec{e}_1) = \vec{e}_1 - \mathrm{Proj}_{\vec{n}}(\vec{e}_1) = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1\cdot\frac{1}{\sqrt{3}}+0\cdot\frac{1}{\sqrt{3}}+0\cdot\frac{1}{\sqrt{3}}}{1}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2/3\\-1/3\\-1/3\end{bmatrix}$$

$$\mathrm{Proj}_{x+y+z=0}(\vec{e}_2) = \vec{e}_2 - \mathrm{Proj}_{\vec{n}}(\vec{e}_2) = \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{0\cdot\frac{1}{\sqrt{3}}+1\cdot\frac{1}{\sqrt{3}}+0\cdot\frac{1}{\sqrt{3}}}{1}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-1/3\\2/3\\-1/3\end{bmatrix}$$

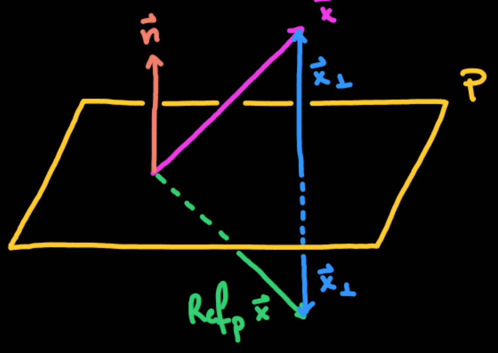
$$\mathrm{Proj}_{x+y+z=0}(\vec{e}_3) = \vec{e}_3 - \mathrm{Proj}_{\vec{n}}(\vec{e}_3) = \begin{bmatrix}0\\0\\1\end{bmatrix} - \frac{0\cdot\frac{1}{\sqrt{3}}+0\cdot\frac{1}{\sqrt{3}}+1\cdot\frac{1}{\sqrt{3}}}{1}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-1/3\\-1/3\\2/3\end{bmatrix}$$

so the matrix is:

$$\begin{bmatrix}\mathrm{Proj}_{x+y+z=0}(\vec{e}_1) & \mathrm{Proj}_{x+y+z=0}(\vec{e}_2) & \mathrm{Proj}_{x+y+z=0}(\vec{e}_3)\end{bmatrix} = \begin{bmatrix}2/3 & -1/3 & -1/3\\-1/3 & 2/3 & -1/3\\-1/3 & -1/3 & 2/3\end{bmatrix}.$$

<u>Example:</u> Find the matrix associated to the reflection across the plane $x+y+z=0$.

Geometrically, we have:



so $\mathrm{Ref}_P(\vec{x}) = \vec{x} - 2\vec{x}_\perp$, where $\vec{x}_\perp$ is the shortest vector from $\vec{x}$ to P.

Using that the unit vector normal to the plane is $\vec{n} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ and knowing

$$\vec{x}_\perp = \mathrm{Proj}_{\vec{n}}(\vec{x}) = \frac{\vec{x}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \frac{1}{3}\begin{bmatrix}x_1+x_2+x_3\\x_1+x_2+x_3\\x_1+x_2+x_3\end{bmatrix},$$ we can compute:

$$\mathrm{Ref}_{x+y+z=0}(\vec{x}) = \vec{x} - 2\vec{x}_\perp = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} - \frac{2}{3}\begin{bmatrix}x_1+x_2+x_3\\x_1+x_2+x_3\\x_1+x_2+x_3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}x_1-2x_2-2x_3\\-2x_1+x_2-2x_3\\-2x_1-2x_2+x_3\end{bmatrix} =$$

$$= \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Since reflections are linear, we could have computed the columns of the matrix as above:

$$\mathrm{Ref}_{x+y+z=0}(\vec{e}_1) = \vec{e}_1 - 2\,\mathrm{Proj}_{\vec{n}}(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

$$\mathrm{Ref}_{x+y+z=0}(\vec{e}_2) = \vec{e}_2 - 2\,\mathrm{Proj}_{\vec{n}}(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$\mathrm{Ref}_{x+y+z=0}(\vec{e}_3) = \vec{e}_3 - 2\,\mathrm{Proj}_{\vec{n}}(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

so that:

$$\begin{bmatrix} \mathrm{Ref}_{x+y+z=0}(\vec{e}_1) & \mathrm{Ref}_{x+y+z=0}(\vec{e}_2) & \mathrm{Ref}_{x+y+z=0}(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}.$$

<u>Example:</u> Find the matrix associated to the linear transformation $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ satisfying:

$$f\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad f\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

We compute its columns:

$$f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = f\left(\frac{1}{2}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)\right) = \frac{1}{2}\left(f\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) + f\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) - f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)\right) =$$

$$= \frac{1}{2}\left(\frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right) = \frac{1}{2}\cdot\frac{1}{3}\begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}.$$

$$f\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=f\left(\frac{1}{2}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}-\begin{bmatrix}1\\0\\1\end{bmatrix}+\begin{bmatrix}0\\1\\1\end{bmatrix}\right)\right)=\frac{1}{2}\left(f\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)-f\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)+f\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)\right)=$$

$$=\frac{1}{2}\left(\frac{1}{3}\begin{bmatrix}1\\1\\-2\end{bmatrix}-\frac{1}{3}\begin{bmatrix}1\\-2\\1\end{bmatrix}+\frac{1}{3}\begin{bmatrix}-2\\1\\1\end{bmatrix}\right)=\frac{1}{2}\cdot\frac{1}{3}\begin{bmatrix}-2\\4\\-2\end{bmatrix}=\begin{bmatrix}-1/3\\2/3\\-1/3\end{bmatrix}.$$

$$f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=f\left(\frac{1}{2}\left(-\begin{bmatrix}1\\1\\0\end{bmatrix}+\begin{bmatrix}1\\0\\1\end{bmatrix}+\begin{bmatrix}0\\1\\1\end{bmatrix}\right)\right)=\frac{1}{2}\left(-f\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)+f\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)+f\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)\right)=$$

$$=\frac{1}{2}\left(-\frac{1}{3}\begin{bmatrix}1\\1\\-2\end{bmatrix}+\frac{1}{3}\begin{bmatrix}1\\-2\\1\end{bmatrix}+\frac{1}{3}\begin{bmatrix}-2\\1\\1\end{bmatrix}\right)=\frac{1}{2}\cdot\frac{1}{3}\begin{bmatrix}-2\\-2\\4\end{bmatrix}=\begin{bmatrix}-1/3\\-1/3\\2/3\end{bmatrix}.$$

The matrix associated to $f:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ is:

$$\left[\;f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)\quad f\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)\quad f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)\;\right]=\begin{bmatrix}2/3&-1/3&-1/3\\-1/3&2/3&-1/3\\-1/3&-1/3&2/3\end{bmatrix},$$

which means that this function is exactly projection onto the plane $x+y+z=0$.