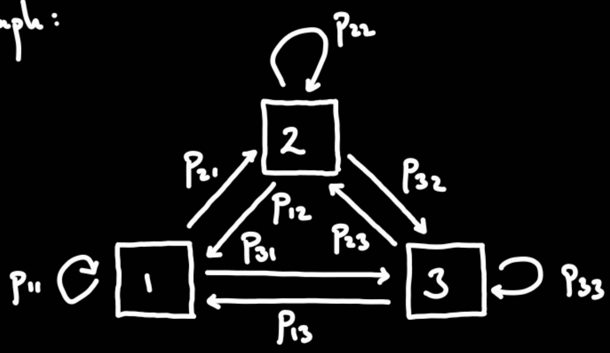
## 4.3. Random walks

We can express the behavior of randomly moving between some predetermined locations by multiplying by a matrix. Given three locations, which we represent as vertices, and the probabilities of moving between them, which we represent as labeled edges, we obtain the following graph:




where the label $p_{ij}$ is the probability of moving from location $j$ to location $i$. In particular, we must have:

$$p_{11}+p_{21}+p_{31}=1\,, \qquad p_{12}+p_{22}+p_{32}=1\,, \qquad p_{13}+p_{23}+p_{33}=1\,,$$

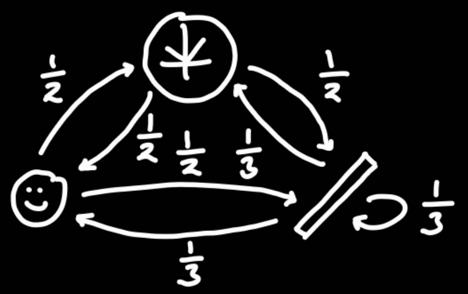
because the probabilities of moving from one location to another is one. Let $x_{01}$, $x_{02}$, $x_{03}$ the probabilities of starting at locations $1, 2, 3$, respectively (at time $t=0$). After one step (so at time $t=1$), the probabilities of being at locations $1, 2, 3$, are:

$$\begin{cases} x_{11} = p_{11}\,x_{01} + p_{12}\,x_{02} + p_{13}\,x_{03}\,, \\ x_{12} = p_{21}\,x_{01} + p_{22}\,x_{02} + p_{23}\,x_{03}\,, \\ x_{13} = p_{31}\,x_{01} + p_{32}\,x_{02} + p_{33}\,x_{03}\,. \end{cases}$$

Writing $\vec{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$, $\vec{x}_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}$, $P = \begin{bmatrix} p_{11} & p_{21} & p_{31} \\ p_{12} & p_{22} & p_{32} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}$, then $\vec{x}_1 = P\,\vec{x}_0$. In general, after n steps (at time $t=n$) we have $\vec{x}_n = P^n \vec{x}_0$, and the relation between taking n steps and n-1 steps is $\vec{x}_n = P\,\vec{x}_{n-1}$.

<u>Example:</u> We are flipping a coin that may land on heads, tails, or its side. It behaves just like a fair three sided coin, except it will never land heads twice in a row and it will never land tails twice in a row. Write the matrix P of the random system. Starting with a coin that landed on heads, find the probability that after three more flips the coin lands on its side. Regardless of the side on which the coin lands before the random process, find the probabilities of the coin landing heads, tails, or its side, after infinitely many flips.

If the coin lands heads, because it never lands heads twice, it will then land tails or side. Since it behaves like a regular coin, each of these options occur with probability $\frac{1}{2}$. Similarly, if the coin lands tails, the next flip it will land heads or side with probabilities $\frac{1}{2}$ each. If the coin lands side, since it behaves like a fair three sided coin, it will then land on heads, tails, or side, with probability $\frac{1}{3}$ each. This gives the following graph:




namely $P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$.

Starting with a coin that landed on heads means $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ so:

$$\vec{x}_3 = P^3 \vec{x}_0 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{25}{72} & \frac{31}{108} \\ \frac{25}{72} & \frac{2}{9} & \frac{31}{108} \\ \frac{31}{72} & \frac{31}{72} & \frac{23}{54} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} \\ \frac{25}{72} \\ \frac{31}{72} \end{bmatrix}$$

meaning that the probability of landing side after three more flips is $\frac{31}{72}$.

Finding the probabilities of the coin landing heads, tails, or its side, after infinitely many flips, amounts to computing the limit $P^\infty = \lim_{n \to \infty} P^n$. This is really hard to do in general, but very easy if we can diagonalize $P$, as we will do in a few weeks.

Diagonalization will tell us that $P = ADA^{-1}$ for $D$ a diagonal matrix. In our case:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -1 & \frac{-1}{2} & \frac{2}{3} \\ 1 & \frac{-1}{2} & \frac{2}{3} \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{2} & 0 & 0 \\ 0 & \frac{-1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{-3}{7} & \frac{-3}{7} & \frac{4}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{3}{7} \end{bmatrix}, \text{ with:}$$

$$\begin{bmatrix} -1 & \frac{-1}{2} & \frac{2}{3} \\ 1 & \frac{-1}{2} & \frac{2}{3} \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{-3}{7} & \frac{-3}{7} & \frac{4}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{-3}{7} & \frac{-3}{7} & \frac{4}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{3}{7} \end{bmatrix} \begin{bmatrix} -1 & \frac{-1}{2} & \frac{2}{3} \\ 1 & \frac{-1}{2} & \frac{2}{3} \\ 0 & 1 & 1 \end{bmatrix},$$

so:

$$P^n = \begin{bmatrix} -1 & \frac{-1}{2} & \frac{2}{3} \\ 1 & \frac{-1}{2} & \frac{2}{3} \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{2} & 0 & 0 \\ 0 & \frac{-1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{-3}{7} & \frac{-3}{7} & \frac{4}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{3}{7} \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & \frac{-1}{2} & \frac{2}{3} \\ 1 & \frac{-1}{2} & \frac{2}{3} \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{2^n} & 0 & 0 \\ 0 & \frac{-1}{6^n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{-3}{7} & \frac{-3}{7} & \frac{4}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{3}{7} \end{bmatrix},$$

and we obtain:

$$P^\infty = \lim_{n\to\infty} P^n = \begin{bmatrix} -1 & \frac{-1}{2} & \frac{2}{3} \\ 1 & \frac{-1}{2} & \frac{2}{3} \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lim_{n\to\infty} \frac{-1}{2^n} & 0 & 0 \\ 0 & \lim_{n\to\infty} \frac{-1}{6^n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{-3}{7} & \frac{-3}{7} & \frac{4}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{3}{7} \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & \frac{-1}{2} & \frac{2}{3} \\ 1 & \frac{-1}{2} & \frac{2}{3} \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{-3}{7} & \frac{-3}{7} & \frac{4}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{3}{7} \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{3}{7} \end{bmatrix}$$

This means that after infinitely many flips, independently of whether we start at head, tails, or side, we will flip heads with probability $\frac{2}{7}$, we will flip tails with probability $\frac{2}{7}$, and we will flip side with probability $\frac{3}{7}$.

This setup can be generalized to admit an arbitrary number of locations $n$, and the $n^2$ probabilities of moving between them can be arranged in an $n \times n$ matrix $P$.

## 4.4. The transpose.

Given an $m\times n$ matrix $A$, its <u>transpose</u> is the $n\times m$ matrix whose $(i,j)$-entry is $a_{ji}$, and we denote it $A^T$. In other words, the $j$-th column of $A^T$ is the $j$-th row of $A$. In other words, $A^T$ is obtained by reflecting $A$ across its diagonal.

<u>Example:</u> Find the transpose of:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}.$$

We can write the dot product $\vec{x}\cdot\vec{y}$ as the matrix multiplication $\vec{x}^T\vec{y}$.

<u>Example:</u> Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Compute $\vec{x}\cdot\vec{y}$ and $\vec{x}^T\vec{y}$.

We have:

$$\vec{x}\cdot\vec{y} = 1\cdot 2 + 2\cdot(-1) + 3\cdot 2 = 6 \quad \text{and} \quad \vec{x}^T\vec{y} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 1\cdot 2 + 2\cdot(-1) + 2\cdot 3 = 6.$$

Conceptually, $A^T$ captures the <u>adjoint</u> of the linear transformation given by $A$. This is encapsulated in the equality: $(A\vec{x})\cdot\vec{y} = \vec{x}\cdot(A^T\vec{y})$

where $A$ is $m\times n$, $\vec{x}$ is $n\times 1$, and $\vec{y}$ is $m\times 1$.

Example: Compute $A\vec{x}$, $A^T\vec{y}$, $(A\vec{x})\cdot\vec{y}$, and $\vec{x}\cdot(A^T\vec{y})$ for:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

We have:

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 19 \\ 31 \end{bmatrix},$$

$$A^T\vec{y} = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix},$$

$$(A\vec{x})\cdot\vec{y} = (A\vec{x})^T\vec{y} = \begin{bmatrix} 7 & 19 & 31 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 7\cdot 1 + 19\cdot(-1) + 31\cdot 1 = 19,$$

$$\vec{x}\cdot(A^T\vec{y}) = \vec{x}^T(A^T\vec{y}) = \begin{bmatrix} 1 & 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 1\cdot 5 + 0\cdot 6 + 2\cdot 7 + 0\cdot 8 = 19.$$

Given $A$ and $B$ matrices of sizes $m\times n$ and $n\times p$, we have: $(AB)^T = B^TA^T$.

We can interpret this by seeing $A$, $A^T$, $B$, $B^T$ as linear transformations $A: \mathbb{R}^n \to \mathbb{R}^m$, $A^T: \mathbb{R}^m \to \mathbb{R}^n$, $B: \mathbb{R}^p \to \mathbb{R}^n$, and $B^T: \mathbb{R}^n \to \mathbb{R}^p$, together with:

$AB : \mathbb{R}^p \xrightarrow{B} \mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$ so the equality $(AB)^T = B^T A^T$ says that

$(AB)^T : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ coincides with $B^T A^T : \mathbb{R}^m \xrightarrow{A^T} \mathbb{R}^n \xrightarrow{B^T} \mathbb{R}^p$.

<u>Example:</u> Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$. Compute $AB$, $(AB)^T$, and $B^T A^T$.

We have:

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 5 \end{bmatrix}.$$

$$(AB)^T = \begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix},$$

$$B^T A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}.$$

## 4.5. Matrix inverses.

Given a square matrix $A$ of size $n \times n$, we say that $A$ is <u>invertible</u> if there is an $n \times n$ matrix $B$ satisfying $AB = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = BA$. When this happens, we say that $B$ is the <u>inverse</u> of $A$, and denote it by $A^{-1}$.

Knowing that a matrix is invertible and knowing how to compute its inverse is useful to solve systems of linear equations. Namely, given a system of equations $A\vec{x} = \vec{b}$ with $A$ invertible, we can multiply by $A^{-1}$ on the left to obtain:

$$A\vec{x} = \vec{b} \longrightarrow A^{-1}A\vec{x} = A^{-1}\vec{b} \longrightarrow \vec{x} = A^{-1}\vec{b}.$$

The matrix $Id_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ is called the $n \times n$ identity matrix, and behaves like the scalar 1:

$r1 = r = 1r$ for all scalars $r$, and

$A\,Id_n = A = Id_n$ for all $n \times n$ matrices $A$.

We can use $Id_n$ to interpret multiplying by the inverse $A^{-1}$ as dividing by $A$:

$r$ non-zero scalar yields $r \cdot \frac{1}{r} = 1 = \frac{1}{r} \cdot r$, and

$A$ invertible matrix yields $A A^{-1} = Id_n = A^{-1}A$.

We can also interpret $A^{-1}$ as the matrix associated to the linear transformation that reverts what the linear transformation given by $A$ does. Namely, $A^{-1}$ undoes $A$:

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n & \xrightarrow{A^{-1}} & \mathbb{R}^n \\ \vec{x} & \longmapsto & A\vec{x} & \longmapsto & A^{-1}A\vec{x} = \vec{x} \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{A^{-1}} & \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \\ \vec{x} & \longmapsto & A^{-1}\vec{x} & \longmapsto & AA^{-1}\vec{x} = \vec{x} \end{array}$$

<u>Example:</u> The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible, with inverse $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$.

**Example:** The matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is <u>not</u> invertible. To see this, suppose there is a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ then } \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ impossible.}$$

There are many equivalent ways of determining when an $n \times n$ matrix $A$ is invertible:

(i) For all $\vec{b}$ in $\mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ has exactly one solution.

(ii) The system $A\vec{x} = \vec{0}$ has exactly one solution (which is $\vec{x} = \vec{0}$).

(iii) The rank of $A$ is $n$.

(iv) The image of $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is all of $\mathbb{R}^n$.

Once we know that a matrix $A$ is invertible, we compute its inverse as follows.

(1) Write an $n \times (2n)$ matrix $[A \mid Id_n]$.

(2) Bring $A$ to its reduced row echelon form, doing the operations on the whole $n \times (2n)$ matrix.

(3) We have $[Id_n \mid B]$, and $B$ is the inverse of $A$.

This method simultaneously computes the rank of $A$, and its inverse (if $A$ is invertible).

**Example:** When is $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible? When it is, compute its inverse.

We reduce the left half of the following matrix.

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow[a \neq 0]{\frac{1}{a}R_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_2 - cR_1}$$

$$\left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \xrightarrow[ad-bc \neq 0]{\frac{a}{ad-bc}R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \xrightarrow{R_1 - \frac{b}{a}R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right].$$

We would like to declare that $A$ is invertible when $ad-bc \neq 0$, in which case

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. However, we also used $a \neq 0$ in the first row operation.

We can check directly that our candidate for $A^{-1}$ always works:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ab \\ cd-dc & -cb+da \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and since we only need $ad-bc \neq 0$ to write $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, then $A$ is

invertible exactly when $ad-bc \neq 0$, in which case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. We can check this with $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, indeed:

$$A^{-1} = \frac{1}{4 \cdot 1 - 2 \cdot 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix},$$ as we saw above.

We now investigate the reason why this computation gives us the inverse. To simplify the argument we only do it for $3 \times 3$ matrices, but the same holds in general.

Given a $3 \times 3$ invertible matrix $A$, its inverse $B$ satisfies $AB\vec{x} = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^3$. Seeing $B : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as a linear transformation, it is represented by the matrix $\begin{bmatrix} B(\vec{e}_1) & B(\vec{e}_2) & B(\vec{e}_3) \end{bmatrix}$. This gives three equalities:

$$A(B(\vec{e}_1)) = \vec{e}_1, \qquad A(B(\vec{e}_2)) = \vec{e}_2, \qquad A(B(\vec{e}_3)) = \vec{e}_3,$$

so we do <u>not</u> know $B(\vec{e}_1)$, $B(\vec{e}_2)$, and $B(\vec{e}_3)$, but we know they are solutions of:

$$A\vec{x} = \vec{e}_1, \qquad A\vec{x} = \vec{e}_2, \qquad A\vec{x} = \vec{e}_3,$$

respectively. Since $A$ is invertible, for each $\vec{b}$ in $\mathbb{R}^3$ the system $A\vec{x} = \vec{b}$ has exactly one solution, so solving the three systems above exactly gives $B(\vec{e}_1)$, $B(\vec{e}_2)$, and $B(\vec{e}_3)$, respectively. For this, we would write:

$$\left[ A \,\middle|\, \vec{e}_1 \right], \quad \left[ A \,\middle|\, \vec{e}_2 \right], \quad \left[ A \,\middle|\, \vec{e}_3 \right],$$

we would reduce the left sides of these augmented matrices, and obtain the solution on the right side. Since $A$ is invertible, its rank is 3, so its reduced row echelon form is $Id_3$. This gives:

$$\left[Id_3 \mid B(\vec{e}_1)\right], \quad \left[Id_3 \mid B(\vec{e}_2)\right], \quad \left[Id_3 \mid B(\vec{e}_3)\right].$$

When reducing $A$ we are always doing the exact same row operations, so to be more efficient we can combine the right sides of the augmented matrix into one:

$$\left[A \mid \vec{e}_1 \ \ \vec{e}_2 \ \ \vec{e}_3\right] \xrightarrow{\text{reduction}} \left[Id_3 \mid B(\vec{e}_1) \ \ B(\vec{e}_2) \ \ B(\vec{e}_3)\right].$$

This is precisely the recipe for computing the inverse of $A$:

$$\left[\begin{array}{c|ccc} A & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{reduction}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & \end{array}\right].$$

Geometrically, $A^{-1}$ undoes the transformation $A$, but they are hard to visualize.

<u>Example:</u> Compute the inverse of a rotation by $\theta$ degrees counterclockwise.

To undo a counterclockwise rotation by $\theta$, we can do a clockwise rotation by $\theta$, or equivalently a counterclockwise rotation by $-\theta$. The counterclockwise rotation is given by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, the clockwise rotation is given by

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$ These are indeed inverses of each other:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then}$$

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} =$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$BA = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} =$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & (-\sin\theta)(-\sin\theta) + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Given $A$ and $B$ invertible $n\times n$ matrices, we have: $(AB)^{-1} = B^{-1}A^{-1}$.

We can interpret this by seeing $A, A^{-1}, B, B^{-1}$ as linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$AB : \mathbb{R}^n \xrightarrow{B} \mathbb{R}^n \xrightarrow{A} \mathbb{R}^n$ so the equality $(AB)^{-1} = B^{-1}A^{-1}$ says that

$(AB)^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ coincides with $B^{-1}A^{-1} : \mathbb{R}^n \xrightarrow{A^{-1}} \mathbb{R}^n \xrightarrow{B^{-1}} \mathbb{R}^n$.

This is similar to what happens for transposes, although more clear conceptually:

$AB$ is doing $B$ then $A$, $B^{-1}A^{-1}$ is undoing $A$ then undoing $B$, and the equality

$(AB)^{-1} = B^{-1}A^{-1}$ says that to undo the function that does $B$ then $A$ we just need to undo $A$ then undo $B$. We can represent these functions as:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "B"] & \mathbb{R}^n \arrow[r, "A"] \arrow[l, bend left, "B^{-1}"] & \mathbb{R}^n \arrow[l, bend left, "A^{-1}"]
\end{tikzcd}$$

namely doing $B$, doing $A$, undoing $A$, undoing $B$, is the same as doing nothing.

Despite these similarities, inverses and transposes do <u>not</u> coincide in general. ⚠

Even more unfortunately, to add to the confusion, <u>sometimes</u> the inverse of a matrix is its transpose. We need to be <u>very</u> careful about these distinctions.

<u>Example:</u> The inverse of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = A^T$.

<u>Example:</u> Compute the inverse of $A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

We have:

$$\left[\begin{array}{ccc|ccc} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 0 & 1 & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 & 1 \end{array}\right] \xrightarrow{3R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & 3 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 0 & 1 & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + \frac{2}{3}R_1]{R_2 + \frac{2}{3}R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -2 & 3 & 0 & 0 \\ 0 & -1 & -2 & 2 & 1 & 0 \\ 0 & -2 & -1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & 3 & 0 & 0 \\ 0 & 1 & 2 & -2 & -1 & 0 \\ 0 & -2 & -1 & 2 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + 2R_2]{R_1 + 2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & -2 & 0 \\ 0 & 1 & 2 & -2 & -1 & 0 \\ 0 & 0 & 3 & -2 & -2 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & -2 & 0 \\ 0 & 1 & 2 & -2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right] \xrightarrow[R_2-2R_3]{R_1-2R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right]$$

Then $A^{-1} = \begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = A^T.$

## 4.6. Determinants.

Recall that we can compute the determinant of a square matrix by expanding along the first row. In fact, we can expand along any row or column, as long as we are careful with the signs. Expanding along the j-th column is:

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = (-1)^{1+j} a_{1j} \cdot \det(\hat{A}_{1j}) + \cdots + (-1)^{i+j} a_{ij} \det(\hat{A}_{ij}) + \cdots + (-1)^{n+j} \det(\hat{A}_{nj})$$

where $\hat{A}_{ij}$ is the $(n-1)\times(n-1)$ matrix obtained when removing the i-th row and j-th column of $A$. Expanding along the i-th row is:

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = (-1)^{i+1} a_{i1} \cdot \det(\hat{A}_{i1}) + \cdots + (-1)^{i+j} a_{ij} \det(\hat{A}_{ij}) + \cdots + (-1)^{i+n} \det(\hat{A}_{in}).$$

<u>Example:</u> Compute the following determinant by expanding along the second row.

$$\det \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix} = (-1)^{2+1} \cdot 3 \cdot \det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + (-1)^{2+2} \cdot 4 \cdot \det \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} + (-1)^{2+3} \cdot 2 \cdot \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} =$$

$$= -3 \cdot (6-1) + 4 \cdot (3-1) - 2 \cdot (1-2) = -5.$$

Compute the following determinant by expanding along the third column:

$$\det \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix} = (-1)^{1+3} \cdot 1 \cdot \det \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} + (-1)^{2+3} \cdot 2 \cdot \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + (-1)^{3+3} \cdot 3 \cdot \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =$$

$$= 1 \cdot (3-4) - 2 \cdot (1-2) + 3 \cdot (4-6) = -5.$$

These expansions can be used to simplify computations, for example in cases when a matrix has many zeros. Another way of computing determinants is via Gaussian elimination, for which it is useful to know how elementary row operations change the determinant of a matrix.

(i) If $B$ is obtained from $A$ by multiplying a row by $r$: $\det(B) = r \cdot \det(A)$.

(ii) If $B$ is obtained from $A$ by swapping two rows: $\det(B) = -\det(A)$.

(iii) If $B$ is obtained from $A$ by adding a row to another row: $\det(B) = \det(A)$.

Example: Compute $\det\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian elimination.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{\frac{-1}{2}R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[R_3 + R_2]{R_1 - 2R_2}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{5}{2} \end{bmatrix} \xrightarrow{\frac{2}{5}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know $\det(Id_3) = 1$, and that we can obtain $B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ from $A = Id_3$

by reversing these row operations, namely:

$$A \xrightarrow{R_2 + \frac{1}{2}R_3;\ \frac{5}{2}R_3;\ R_3 - R_2;\ R_1 + 2R_2;\ -2R_2;\ R_3 + R_1;\ R_2 - 3R_1} B$$

so we obtain:

$$\det(B) = \frac{5}{2} \cdot (-2) \cdot \det(A) = -5.$$

This behavior on row operations gives the following properties:

(i) $A$ is invertible if and only if $\det(A) \neq 0$.

(ii) $\det(AB) = \det(A)\det(B)$.

(iii) $\det(A^T) = \det(A)$.

(iv) $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Moreover, if we have matrices whose only non-zero entries are on the diagonal and above, or on the diagonal and below, their determinant is the product of their diagonal entries. For example, in the case of 5x5 matrices we have the following.

$$\det\begin{bmatrix} a_{11} & & & & \\ & a_{22} & & * & \\ & & a_{33} & & \\ & 0 & & a_{44} & \\ & & & & a_{55} \end{bmatrix} = a_{11}\cdot a_{22}\cdot a_{33}\cdot a_{44}\cdot a_{55} = \det\begin{bmatrix} a_{11} & & & & \\ & a_{22} & & & 0 \\ & & a_{33} & & \\ & * & & a_{44} & \\ & & & & a_{55} \end{bmatrix}.$$

So the determinant of a matrix is the multiplication of the diagonal entries of its row echelon form, and the sign is given by the number of times that two rows were swapped:

$$\det(A) = (-1)^{\text{number of swaps to obtain REF}} \cdot (\text{product of diagonal entries of REF}).$$

Example: Compute $\det\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ using its row echelon form.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 - \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}$$

So: $\det\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix} = 1\cdot(-2)\cdot\left(\frac{5}{2}\right)\cdot(-1)^0 = -5.$

A formal definition of the determinant is the unique function $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ having $n\times n$ matrices for input and scalars for output, which is also linear on each column, alternating on columns, linear on each row, alternating on rows, and $\det(Id_n) = 1$.

This means:

(i) linearity on columns:

$$\det\begin{bmatrix} c_1 & \cdots & c_j + c_j' & \cdots & c_n \end{bmatrix} = \det\begin{bmatrix} c_1 \cdots c_j \cdots c_n \end{bmatrix} + \det\begin{bmatrix} c_1 \cdots c_j' \cdots c_n \end{bmatrix}.$$

$$\det\begin{bmatrix} c_1 & \cdots & r\cdot c_j & \cdots & c_n \end{bmatrix} = r\cdot\det\begin{bmatrix} c_1 \cdots c_j \cdots c_n \end{bmatrix}.$$

(ii) alternating on columns:

$$\det\begin{bmatrix} c_1 \cdots c_j \cdots c_j \cdots c_n \end{bmatrix} = 0.$$

(iii) linear on rows:

$$\det\begin{bmatrix} R_1 \\ \vdots \\ R_i + R_i' \\ \vdots \\ R_n \end{bmatrix} = \det\begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_n \end{bmatrix} + \det\begin{bmatrix} R_1 \\ \vdots \\ R_i' \\ \vdots \\ R_n \end{bmatrix}$$

$$\det\begin{bmatrix} R_1 \\ \vdots \\ r\cdot R_i \\ \vdots \\ R_n \end{bmatrix} = r\cdot\det\begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_n \end{bmatrix}$$

(iv) alternating on rows:

$$\det\begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_i \\ \vdots \\ R_n \end{bmatrix} = 0.$$

(v) the identity has determinant one:

$$\det\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = 1.$$

As a consequence of linearity and alternating we have:

$$\det\begin{bmatrix} C_1 \cdots C_i \cdots C_j \cdots C_n \end{bmatrix} = -\det\begin{bmatrix} C_1 \cdots C_j \cdots C_i \cdots C_n \end{bmatrix},$$

$$\det\begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_n \end{bmatrix} = -\det\begin{bmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ R_i \\ \vdots \\ R_n \end{bmatrix}.$$

<u>Example:</u> Compute $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ using the formal definition of the determinant.

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \det\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} =$$

$$= a\det\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} + c\det\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix} =$$

$$= a\left(\det\begin{bmatrix}1 & b\\0 & 0\end{bmatrix} + \det\begin{bmatrix}1 & 0\\0 & d\end{bmatrix}\right) + c\left(\det\begin{bmatrix}0 & b\\1 & 0\end{bmatrix} + \det\begin{bmatrix}0 & 0\\1 & d\end{bmatrix}\right) =$$

$$= ab\det\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + ad\det\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + cb\det\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + cd\det\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix} =$$

$$= ad\det\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - cb\det\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = ad - bc.$$

The geometric meaning of the determinant is the volume of the parallelotope determined by its columns. All of its properties can be derived from this interpretation, but they are hard to visualize. Given $A$ an $n\times n$ matrix, seen as a linear transformation $A:\mathbb{R}^n \longrightarrow \mathbb{R}^n$, the determinant of $A$ measures the ratio between a volume in the input and the volume after undergoing the linear transformation $A$. That is, given a geometric figure in $\mathbb{R}^n$ with volume $V$, applying $A$ to each point of the figure will give a geometric figure in $\mathbb{R}^n$ with volume $|\det(A)|\cdot V$.

<u>Example:</u> The triangle with vertices $(-1,0), (1,1), (0,1)$ has area $1$. After applying

$$A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix},$$ it becomes the triangle with vertices $(-1,-3), (3,7), (2,4)$,

which has area $$\left|\det\begin{bmatrix}4 & 3\\10 & 7\end{bmatrix}\right| = 2 = \left|\det\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\right|\cdot 1.$$