As we saw, as soon as a matrix A has one eigenvector, it has many eigenvectors: if $A\vec{v} = \lambda\vec{v}$ then $A(s\vec{v}) = \lambda(s\vec{v})$ for all scalars s. Thankfully, they all share the same eigenvalue as the original eigenvector. This tells us that eigenvectors are <u>not</u> that important themselves, what is more important is the direction they point towards. This direction and the eigenvalue associated to it is what matters. To compute them, we note that if $\vec{v}$ is an eigenvector with eigenvalue λ, then $A\vec{v} = \lambda\vec{v}$ means that $A\vec{v} - \lambda\vec{v} = \vec{0}$ so $(A - \lambda \cdot \mathrm{Id}_n)\vec{v} = \vec{0}$. Since $\vec{v}$ is an eigenvector then $\vec{v} \neq \vec{0}$, meaning that $(A - \lambda \cdot \mathrm{Id}_n)\vec{x} = \vec{0}$ has at least two solutions, namely $\vec{x} = \vec{0}$ and $\vec{x} = \vec{v}$. Since $(A - \lambda \cdot \mathrm{Id}_n)\vec{x} = \vec{0}$ has more than one solution, the matrix $A - \lambda \cdot \mathrm{Id}_n$ is <u>not</u> invertible, so $\det(A - \lambda \cdot \mathrm{Id}_n) = 0$. Now $\det(A - \lambda \cdot \mathrm{Id}_n)$ is a polynomial, we can factor it into linear terms to find its roots, and those roots are the eigenvalues of A. Once we know the eigenvalues of A, for each eigenvalue λ we solve the equation $(A - \lambda \cdot \mathrm{Id}_n)\vec{x} = \vec{0}$ for $\vec{x}$. Those solutions will be the eigenvectors of A with eigenvalue λ. We can break this process down to the following steps:

1. Compute the <u>characteristic polynomial</u> $\det(A - \lambda \cdot \mathrm{Id}_n)$.

② Find the roots $\lambda_1, \dots, \lambda_r$ of $\det(A - \lambda \cdot Id_n)$, these are the eigenvalues of A.

③ Find the non-zero solutions of $(A - \lambda_1 \cdot Id_n)\vec{x}_1 = 0, \dots, (A - \lambda_r \cdot Id_n)\vec{x}_r = \vec{0}$,

these are the eigenvectors of A.

<u>Example:</u> Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

We have:

$$\det(A - \lambda \cdot Id_2) = \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 =$$

$$= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1).$$

$$\lambda = \frac{-(-2) \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{2 \pm 4}{2} = \begin{cases} \frac{6}{2} = 3. \\ \frac{-2}{2} = -1. \end{cases}$$

For $\lambda_1 = 3$, we solve:

$$\vec{0} = (A - \lambda_1 \cdot Id_2)\vec{x} = \begin{bmatrix} 1-\lambda_1 & 2 \\ 2 & 1-\lambda_1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} \vec{x} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \vec{x}, \text{ namely}$$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad \vec{x} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -1$, we solve:

$$\vec{0} = (A - \lambda_2 \cdot Id_2)\vec{x} = \begin{bmatrix} 1-\lambda_2 & 2 \\ 2 & 1-\lambda_2 \end{bmatrix} \vec{x} = \begin{bmatrix} 1+1 & 2 \\ 2 & 1+1 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \vec{x}, \text{ namely}$$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad \vec{x} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \; \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The matrix has two pairs of eigenvectors and eigenvalues:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \lambda_1 = 3 \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \lambda_2 = -1.$$

<u>Example:</u> Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

We have:

$$\det(A - \lambda \cdot Id_2) = \det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = (\lambda - i)(\lambda + i).$$

$$\lambda = \frac{-0 \pm \sqrt{0 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{\pm 2 \cdot \sqrt{-1}}{2} = \begin{cases} +i. \\ -i. \end{cases}$$

In particular, there are <u>no</u> real eigenvalues. For $\lambda_1 = i$, we solve:

$$\vec{0} = (A - \lambda_1 \cdot Id_2)\vec{x} = \begin{bmatrix} -\lambda_1 & -1 \\ 1 & -\lambda_1 \end{bmatrix} \vec{x} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \vec{x}, \text{ namely}$$

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad \vec{x} = s \begin{bmatrix} i \\ 1 \end{bmatrix}, \ \vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -i$, we solve:

$$\vec{0} = (A - \lambda_2 \cdot Id_2)\vec{x} = \begin{bmatrix} -\lambda_2 & -1 \\ 1 & -\lambda_2 \end{bmatrix} \vec{x} = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \vec{x}, \text{ namely}$$

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad \vec{x} = s \begin{bmatrix} -i \\ 1 \end{bmatrix}, \ \vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

The matrix has two pairs of eigenvectors and eigenvalues:

$$\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \lambda_1 = i \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \lambda_2 = -i.$$

<u>Example:</u> Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

We have:

$$\det(A - \lambda \cdot Id_3) = \det\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \lambda \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix} =$$

$$= (2-\lambda)\cdot\left((2-\lambda)^2 - 1\right) - \left((2-\lambda) - 1\right) + \left(1 - (2-\lambda)\right) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4.$$

Since the polynomial has integer entries, we check the divisors of the constant term for roots of the polynomial. The divisors of 4 are $\pm 4, \pm 2, \pm 1$, and if we try $+1$ we find: $-(1)^3 + 6\cdot(1)^2 - 9\cdot(1) + 4 = 0$. So $\lambda - 1$ divides $-\lambda^3 + 6\lambda^2 - 9\lambda + 4$, and doing long division:

$$\begin{array}{l|l}
-\lambda^3 + 6\lambda^2 - 9\lambda + 4 & \lambda - 1 \\
\cline{2-2}
\underline{-\lambda^3 + \lambda^2} & -\lambda^2 + 5\lambda - 4 \\
0 \quad 5\lambda^2 - 9\lambda + 4 & \\
\quad\ \ \underline{5\lambda^2 - 5\lambda} & \\
\quad\ \ 0 \quad -4\lambda + 4 & \\
\qquad\quad\ \ \underline{-4\lambda + 4} & \\
\qquad\qquad 0 \quad 0 &
\end{array}$$

so $-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = (\lambda - 1)(-\lambda^2 + 5\lambda - 4)$.

We repeat the process for $-\lambda^2 + 5\lambda - 4$, obtaining: $-(1)^2 + 5\cdot(1) - 4 = 0$. The division:

$$\begin{array}{l|l}
-\lambda^2 + 5\lambda - 4 & \lambda - 1 \\
\cline{2-2}
\underline{-\lambda^2 + \lambda} & -\lambda + 4 \\
0 \quad 4\lambda - 4 & \\
\quad\ \ \underline{4\lambda - 4} & \\
\quad\ \ \ 0 \quad 0 &
\end{array}$$

gives $-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = (\lambda - 1)(\lambda - 1)(-\lambda + 4)$.

The roots are then 1, 1, and 4. For $\lambda_1 = 1$ and $\lambda_2 = 1$, we solve:

$$\vec{0} = (A - \lambda_1 \cdot Id_3)\vec{x} = \begin{bmatrix} 2-\lambda_1 & 1 & 1 \\ 1 & 2-\lambda_1 & 1 \\ 1 & 1 & 2-\lambda_1 \end{bmatrix}\vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\vec{x}, \text{ namely}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad \vec{x} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and we}$$

have two linearly independent eigenvectors $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda_3 = 4$, we solve:

$$\vec{0} = (A - \lambda_3 \cdot Id_3)\vec{x} = \begin{bmatrix} 2-\lambda_3 & 1 & 1 \\ 1 & 2-\lambda_3 & 1 \\ 1 & 1 & 2-\lambda_3 \end{bmatrix}\vec{x} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\vec{x}, \text{ namely}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad \vec{x} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The matrix has three pairs of eigenvectors and eigenvalues:

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \lambda_1 = 1 \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \lambda_2 = 1 \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \lambda_3 = 4.$$

In all the above examples, there are as many eigenvalues (counted with multiplicity) as the size of the matrix, and when the eigenvectors have real entries they form a basis of the ambient space. More specifically, the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has two pairs of eigenvalues and eigenvectors, and the two eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are a basis of $\mathbb{R}^2$. Similarly, the 3×3 matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ has three pairs of eigenvalues and eigenvectors, and the eigenvectors $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are a basis of $\mathbb{R}^3$.

This is the best case scenario, but it does <u>not</u> always happen.

<u>Example:</u> Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

We have:

$$\det(A - \lambda \cdot Id_2) = \det\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = (\lambda-1)(\lambda-1).$$

For $\lambda_1 = 1$ and $\lambda_2 = 1$, we solve:

$$\vec{0} = (A - \lambda_1 \cdot Id_2)\vec{x} = \begin{bmatrix} 1-\lambda_1 & 1 \\ 0 & 1-\lambda_1 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\vec{x}, \text{ namely}$$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad \vec{x} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Although we have two eigenvectors (counted with multiplicity), there is only one linearly independent eigenvector.

In this course we will deal with matrices where the multiplicity of an eigenvalue coincides with the number of associated linearly independent eigenvectors, but in nature it is common to find matrices where this does not happen.

6.2. Eigenanalysis simplifies matrix powers.

When the eigenvectors of a matrix form a basis of the ambient space, that is when the multiplicity of an eigenvalue coincides with the number of associated linearly independent eigenvectors with real entries, we can use these eigenvectors to simplify the multiplication by that matrix.

Concretely, let A be an $n \times n$ matrix with real eigenvalues $\lambda_1, \dots, \lambda_n$ whose corresponding real eigenvectors are $\vec{v}_1, \dots, \vec{v}_n$, and suppose $\vec{v}_1, \dots, \vec{v}_n$ are a basis of $\mathbb{R}^n$. Recall that this means that we can write any vector $\vec{x}$ in $\mathbb{R}^n$ as a linear combination:

$$\vec{x} = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n$$

where $a_1, \dots, a_n$ are scalars in $\mathbb{R}$. Then multiplication by A on the left gives:

$$A\vec{x} = A(a_1\vec{v}_1 + \dots + a_n\vec{v}_n) = A(a_1\vec{v}_1) + \dots + A(a_n\vec{v}_n) = a_1 A\vec{v}_1 + \dots + a_n A\vec{v}_n =$$

$$= a_1\lambda_1\vec{v}_1 + \dots + a_n\lambda_n\vec{v}_n = \lambda_1 a_1 \vec{v}_1 + \dots + \lambda_n a_n \vec{v}_n .$$

In other words, if we know how to write $\vec{x}$ as a linear combination of $\vec{v}_1, \dots, \vec{v}_n$

with respective scalars $a_1, \dots, a_n$, then $A\vec{x}$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_n$ with respective scalars $\lambda_1 a_1, \dots, \lambda_n a_n$.

<u>Example:</u> Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$, compute $A\vec{x}$, $A^{10}\vec{x}$, and $A^{100}\vec{x}$.

We know that A has two pairs of eigenvector and eigenvalue:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \lambda_1 = 3 \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \lambda_2 = -1.$$

To find scalars a_1 and a_2 such that $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2$ we solve:

$$\begin{bmatrix} 8 \\ 3 \end{bmatrix} = \vec{x} = s\vec{v}_1 + t\vec{v}_2 = s\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{with augmented matrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 8 \\ 1 & 1 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & \frac{11}{2} \\ 0 & 1 & \frac{-5}{2} \end{array}\right] \quad \text{so} \quad \begin{bmatrix} 8 \\ 3 \end{bmatrix} = \frac{11}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then:

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\left(\frac{11}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \frac{11}{2}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} =$$

$$= \frac{11}{2} \cdot 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{5}{2} \cdot (-1) \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 19 \end{bmatrix}.$$

$$A^{10}\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{10}\left(\frac{11}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \frac{11}{2}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{10}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{10}\begin{bmatrix} -1 \\ 1 \end{bmatrix} =$$

$$= \frac{11}{2}\cdot(3^{10})\cdot\begin{bmatrix}1\\1\end{bmatrix} - \frac{5}{2}\cdot(-1)^{10}\cdot\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}\frac{11\cdot 3^{10}+5}{2}\\ \frac{11\cdot 3^{10}-5}{2}\end{bmatrix} = \begin{bmatrix}324772\\324767\end{bmatrix}.$$

$$A^{100}\vec{x} = \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}^{100}\left(\frac{11}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{5}{2}\begin{bmatrix}-1\\1\end{bmatrix}\right) = \frac{11}{2}\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}^{100}\begin{bmatrix}1\\1\end{bmatrix} - \frac{5}{2}\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}^{100}\begin{bmatrix}-1\\1\end{bmatrix} =$$

$$= \frac{11}{2}\cdot 3^{100}\cdot\begin{bmatrix}1\\1\end{bmatrix} - \frac{5}{2}\cdot(-1)^{100}\cdot\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}\frac{11\cdot 3^{100}+5}{2}\\ \frac{11\cdot 3^{100}+5}{2}\end{bmatrix}.$$

This will be particularly useful when working with random walks.

Example: Let $P = \begin{bmatrix}0 & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & 0 & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3}\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, compute $P^{10}\vec{y}$, $P^{100}\vec{y}$, and $\lim_{n\to\infty} P^n\vec{y}$.

Note that P is the matrix of a random walk. We first find the eigenvalues of P:

$$\det(P-\lambda\cdot \mathrm{Id}_3) = \det\begin{bmatrix}-\lambda & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & -\lambda & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3}-\lambda\end{bmatrix} = -\lambda^3 + \frac{1}{3}\lambda^2 + \frac{7}{12}\lambda + \frac{1}{12}$$

and a matrix associated to a random walk always has $\lambda_1 = 1$ as an eigenvalue (because P^T has $\vec{v}_1 = \vec{e}_1 + \cdots + \vec{e}_n$ as an eigenvector of eigenvalue $\lambda_1 = 1$ and $\det(P^T - \lambda\cdot\mathrm{Id}_n) =$ $= \det(P^T - \lambda\cdot \mathrm{Id}_n^T) = \det((P^T - \lambda\cdot\mathrm{Id})^T) = \det(P-\lambda\cdot\mathrm{Id}_n)$, so the characteristic polynomials of P^T and P coincide, so their roots coincide, so their eigenvalues coincide) so we can factor:

$$-\lambda^3 + \frac{1}{3}\lambda^2 + \frac{7}{12}x + \frac{1}{2} = (\lambda-1)\left(-\lambda^2 - \frac{2}{3}\lambda - \frac{1}{12}\right) \quad \text{via the long division}$$

$$
\begin{array}{l|l}
-\lambda^3+\frac{1}{3}\lambda^2+\frac{7}{12}\lambda+\frac{1}{12} & \lambda-1 \\ \cline{2-2}
-\lambda^3+\lambda^2 & -\lambda^2-\frac{2}{3}\lambda-\frac{1}{12} \\ \cline{1-1}
0\quad -\frac{2}{3}\lambda^2+\frac{7}{12}\lambda+\frac{1}{12} & \\
\quad\;\, -\frac{2}{3}\lambda^2+\frac{2}{3}\lambda & \\ \cline{1-1}
\quad\;\, 0\quad -\frac{1}{12}\lambda+\frac{1}{12} & \\
\qquad\qquad -\frac{1}{12}\lambda+\frac{1}{12} & \\ \cline{1-1}
\qquad\qquad\quad 0 \qquad 0 &
\end{array}
$$

Using the quadratic formula we obtain that $-\lambda^2-\frac{2}{3}\lambda-\frac{1}{12}$ has roots $-\frac{1}{2}$ and $-\frac{1}{6}$.

$$
\lambda=\frac{-\left(-\frac{2}{3}\right)\pm\sqrt{\left(-\frac{2}{3}\right)^2-4\cdot(-1)\cdot\left(-\frac{1}{12}\right)}}{2\cdot(-1)}=\frac{\frac{2}{3}\pm\sqrt{\frac{4}{9}-\frac{4}{12}}}{-2}=\frac{\frac{2}{3}\pm\sqrt{\frac{1}{9}}}{-2}=\frac{\frac{2}{3}\pm\frac{1}{3}}{-2}=\begin{cases}\dfrac{\frac{2}{3}+\frac{1}{3}}{-2}=-\frac{1}{2}\\[2ex] \dfrac{\frac{2}{3}-\frac{1}{3}}{-2}=-\frac{1}{6}\end{cases}
$$

Then $-\lambda^2-\frac{2}{3}\lambda-\frac{1}{12}=-\left(\lambda+\frac{1}{2}\right)\left(\lambda+\frac{1}{6}\right)$, so:

$$
-\lambda^3+\frac{1}{3}\lambda^2+\frac{7}{12}x+\frac{1}{2}=-(\lambda-1)\left(\lambda+\frac{1}{2}\right)\left(\lambda+\frac{1}{6}\right)
$$

and we have three eigenvalues $\lambda_1=1$, $\lambda_2=-\frac{1}{2}$, $\lambda_3=-\frac{1}{12}$. We now find their corresponding eigenvectors, for what we solve $(P-\lambda\cdot Id_3)\vec{x}=\vec{0}$ for each of $\lambda=\lambda_1,\lambda_2,\lambda_3$. For $\lambda_1=1$:

$$
\vec{0}=\begin{bmatrix}-1&\frac{1}{2}&\frac{1}{3}\\ \frac{1}{2}&-1&\frac{1}{3}\\ \frac{1}{2}&\frac{1}{2}&\frac{-2}{3}\end{bmatrix}\vec{x}\xrightarrow{\text{augment}}\left[\begin{array}{ccc|c}-1&\frac{1}{2}&\frac{1}{3}&0\\ \frac{1}{2}&-1&\frac{1}{3}&0\\ \frac{1}{2}&\frac{1}{2}&\frac{-2}{3}&0\end{array}\right]\xrightarrow{\text{RREF}}\left[\begin{array}{ccc|c}1&0&\frac{-2}{3}&0\\ 0&1&\frac{-2}{3}&0\\ 0&0&0&0\end{array}\right]
$$

$$
\text{so}\quad \vec{x}=s\begin{bmatrix}\frac{2}{3}\\ \frac{2}{3}\\ 1\end{bmatrix}\quad\text{and}\quad \vec{v}_1=\begin{bmatrix}\frac{2}{3}\\ \frac{2}{3}\\ 1\end{bmatrix}.
$$

For $\lambda_2=-\frac{1}{2}$:

$$
\vec{0}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\ \frac{1}{2}&\frac{1}{2}&\frac{5}{6}\end{bmatrix}\vec{x}\xrightarrow{\text{augment}}\left[\begin{array}{ccc|c}\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\ \frac{1}{2}&\frac{1}{2}&\frac{5}{6}&0\end{array}\right]\xrightarrow{\text{RREF}}\left[\begin{array}{ccc|c}1&1&0&0\\ 0&0&1&0\\ 0&0&0&0\end{array}\right]
$$

so $\vec{x} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda_3 = -\frac{1}{6}$:

$$\vec{0} = \begin{bmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{x} \xrightarrow{\text{augment}} \left[\begin{array}{ccc|c} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $\vec{x} = s\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$.

The eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent (because a collection of eigenvectors with pairwise distinct eigenvalues is always linearly independent), so being three, they form a basis of $\mathbb{R}^3$. This means that we can write:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{y} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = a_1\begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} + a_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + a_3\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

for some scalars a_1, a_2, a_3, for which we solve:

$$\left[\begin{array}{ccc|c} \frac{2}{3} & -1 & \frac{-1}{2} & 0 \\ \frac{2}{3} & 1 & \frac{-1}{2} & 1 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{7} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{7} \end{array}\right] \quad \text{so} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{3}{7}\begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{7}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Now we compute:

$$P^{10}\vec{y} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}^{10} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}^{10} \left(\frac{3}{7}\begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{7}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right) =$$

$$= \frac{3}{7}\cdot(1)^{10}\begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} + \frac{1}{2}\cdot\left(-\frac{1}{2}\right)^{10}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{7}\cdot\left(\frac{-1}{6}\right)^{10}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} - \frac{1}{2^{11}} + \frac{3}{14\cdot 6^{10}} \\ \frac{2}{7} + \frac{1}{2^{11}} + \frac{3}{14\cdot 6^{10}} \\ \frac{3}{7} - \frac{3}{7\cdot 6^{10}} \end{bmatrix}.$$

$$P^{100}\vec{y} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}^{100} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}^{100} \left(\frac{3}{7} \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{7} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right) =$$

$$= \frac{3}{7} \cdot (1)^{100} \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} + \frac{1}{2} \cdot \left(-\frac{1}{2}\right)^{100} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{7} \cdot \left(\frac{-1}{6}\right)^{100} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} - \frac{1}{2^{101}} + \frac{3}{14 \cdot 6^{100}} \\ \frac{2}{7} + \frac{1}{2^{101}} + \frac{3}{14 \cdot 6^{100}} \\ \frac{3}{7} - \frac{3}{7 \cdot 6^{100}} \end{bmatrix}.$$

$$\lim_{n \to \infty} P^n \vec{y} = \lim_{n \to \infty} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}^{n} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \lim_{n \to \infty} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}^{n} \left(\frac{3}{7} \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{7} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right) =$$

$$= \lim_{n \to \infty} \left(\frac{3}{7} \cdot (1)^{n} \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} + \frac{1}{2} \cdot \left(-\frac{1}{2}\right)^{n} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{7} \cdot \left(\frac{-1}{6}\right)^{n} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right) =$$

$$= \lim_{n \to \infty} \left(\frac{3}{7} \cdot (1)^{n} \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} \right) + \lim_{n \to \infty} \left(\frac{1}{2} \cdot \left(-\frac{1}{2}\right)^{n} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) - \lim_{n \to \infty} \left(\frac{3}{7} \cdot \left(\frac{-1}{6}\right)^{n} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right) =$$

$$= \frac{3}{7} \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} + \frac{1}{2} \cdot 0 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{7} \cdot 0 \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ \frac{2}{7} \\ \frac{3}{7} \end{bmatrix}.$$