6.5.1. Diagonalization

The easiest matrices to work with are diagonal matrices. Everything that we have done so far is almost immediate when we are working with a diagonal matrix.

<u>Example:</u> Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

(i) Compute $\det(A)$.

(ii) Compute the reduced row echelon form of A.

(iii) Describe the linear transformation given by A geometrically.

(iv) Compute $\operatorname{rank}(A)$.

(v) Compute A^T.

(vi) If A is invertible, compute A^{-1}.

(vii) Compute A^n for all natural numbers n.

(viii) If A is invertible, compute A^{-n} for all natural numbers n.

(ix) Compute the eigenvalues of A.

(x) Compute the eigenvectors of A.

(i) $\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. This is the multiplication of the diagonal entries of A.

(ii) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This is just dividing each row by its diagonal entry. If a diagonal entry is zero, that row is moved to the bottom.

(iii) The matrix A is rescaling each standard basis vector by a factor. It dilates $\vec{e}_1$ by 1, $\vec{e}_2$ by 2, $\vec{e}_3$ by 3, and $\vec{e}_4$ by 4. This is scaling each standard basis element $\vec{e}_i$ by the diagonal entry in the i-th row.

(iv) $\operatorname{rank}\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}\right) = 4$. This is just the number of non-zero diagonal entries.

(v) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. A diagonal matrix is its own transpose.

(vi) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$. If it exists, the inverse of a diagonal matrix is also a diagonal matrix, and its non-zero entries are the multiplicative inverses of the non-zero entries of the original matrix, in order:

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda_n} \end{bmatrix}.$$

(vii) $\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{bmatrix}^n = \begin{bmatrix}1^n&0&0&0\\0&2^n&0&0\\0&0&3^n&0\\0&0&0&4^n\end{bmatrix}$. This is just the powers of the diagonal entries.

(viii) $\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{bmatrix}^{-n} = \begin{bmatrix}1&0&0&0\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{3}&0\\0&0&0&\frac{1}{4}\end{bmatrix}^n = \begin{bmatrix}1^n&0&0&0\\0&\frac{1}{2^n}&0&0\\0&0&\frac{1}{3^n}&0\\0&0&0&\frac{1}{4^n}\end{bmatrix}$.

(ix) The eigenvalues of $\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{bmatrix}$ are the roots of $\det\begin{bmatrix}1-\lambda&0&0&0\\0&2-\lambda&0&0\\0&0&3-\lambda&0\\0&0&0&4-\lambda\end{bmatrix} =$

$= (1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda)$. These are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, and $\lambda_4 = 4$. The

eigenvalues of a diagonal matrix are exactly its diagonal entries.

(x) For each of $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, and $\lambda_4 = 4$ we solve $\begin{bmatrix}1-\lambda&0&0&0\\0&2-\lambda&0&0\\0&0&3-\lambda&0\\0&0&0&4-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

For $\lambda = 1$ we have the augmented matrix $\left[\begin{array}{cccc|c}0&0&0&0&0\\0&1&0&0&0\\0&0&2&0&0\\0&0&0&3&0\end{array}\right]$ reducing to $\left[\begin{array}{cccc|c}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{array}\right]$

so $\vec{v}_1 = \vec{e}_1$. For $\lambda = 2$ we have the augmented matrix $\left[\begin{array}{cccc|c}-1&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&2&0\end{array}\right]$ reducing to

$\left[\begin{array}{cccc|c}1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{array}\right]$ so $\vec{v}_2 = \vec{e}_2$. For $\lambda = 3$ we have the augmented matrix $\left[\begin{array}{cccc|c}-2&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{array}\right]$

reducing to $\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{array}\right]$ so $\vec{v}_3 = \vec{e}_3$. For $\lambda = 4$ we have the augmented matrix

$\left[\begin{array}{cccc|c}-3&0&0&0&0\\0&-2&0&0&0\\0&0&-1&0&0\\0&0&0&0&0\end{array}\right]$ reducing to $\left[\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\end{array}\right]$ so $\vec{v}_4 = \vec{e}_4$. The eigenvectors of a diagonal matrix are always $\vec{e}_1, \dots, \vec{e}_n$, and their corresponding eigenvalues are $\lambda_1, \dots, \lambda_n$.

Working with a general matrix is <u>not</u> so convenient, but it will be for the following specific family of matrices. A matrix A is called <u>diagonalizable</u> if there exists an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$. An $m \times m$ matrix A is diagonalizable when its eigenvectors form a basis of $\mathbb{R}^m$. A technical distinction is that we need to specify if we are diagonalizable with scalars in $\mathbb{R}$ or $\mathbb{C}$, and although we illustrate this below, it will <u>not</u> be an issue in this course.

<u>Example:</u> Determine which of the following matrices are diagonalizable.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalue-eigenvector pairs of these matrices are:

A: $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $\left(-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$. The eigenvectors form a basis of $\mathbb{R}^2$.

B: $\left(2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $\left(0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$. The eigenvectors form a basis of $\mathbb{R}^2$.

C: $\left(1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. The eigenvectors do <u>not</u> form a basis of $\mathbb{R}^2$.

D: $\left(i, \begin{bmatrix} i \\ 1 \end{bmatrix}\right)$ and $\left(-i, \begin{bmatrix} -i \\ 1 \end{bmatrix}\right)$. The eigenvectors do <u>not</u> form a basis of $\mathbb{R}^2$, but they form a basis of $\mathbb{C}^2$.

This means that A and B are diagonalizable (over $\mathbb{R}$), C and D are <u>not</u> diagonalizable (over $\mathbb{R}$), but if we are working with the complex numbers as scalars then D is diagonalizable (over $\mathbb{C}$).

<u>Example:</u> Determine which of the following matrices are diagonalizable.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalue-eigenvector pairs of these matrices are:

A: $\left(1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right), \left(1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right), \left(4, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$. The eigenvectors form a basis of $\mathbb{R}^3$.

B: $\left(1, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix}\right), \left(-\frac{1}{2}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right), \left(-\frac{1}{6}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right)$. The eigenvectors form a basis of $\mathbb{R}^3$.

C: $\left(1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \left(1, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$. The eigenvectors do <u>not</u> form a basis of $\mathbb{R}^3$.

D: $\left(2, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \left(i, \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}\right), \left(-i, \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}\right)$. The eigenvectors do <u>not</u> form a basis of $\mathbb{R}^3$, but they form a basis of $\mathbb{C}^3$.

This means that A and B are diagonalizable (over $\mathbb{R}$), C and D are <u>not</u> diagonalizable (over $\mathbb{R}$), but if we are working with the complex numbers as scalars then D is diagonalizable (over $\mathbb{C}$).

We can determine when a matrix is diagonalizable by checking that each eigenvalue has as many associated linearly independent eigenvectors as its multiplicity as a root of the characteristic polynomial. This characterization is useful for large matrices, but not so much for the small ones that we will be dealing with.

When the eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ of A form a basis of $\mathbb{R}^n$, the matrix:

$$S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$$

whose columns are precisely the eigenbasis of A is invertible, and the matrix:

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

whose entries are all zero except in the diagonal, where it has the eigenvalues $\lambda_1, \dots, \lambda_n$ of $\vec{v}_1, \dots, \vec{v}_n$ in precisely that order, satisfy $A = SDS^{-1}$. Given a diagonalizable matrix A, the process of finding S and D is known as a <u>diagonalization</u> of A.

<u>Example:</u> Finalize diagonalizing the following matrices given these eigenvalue-eigenvector pairs.

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, $\left(-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$. Consider $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$,

then $S^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$ and $SDS^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} =$

$= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = A$. This diagonalization is <u>not</u> unique, changing the order of the

eigenvectors gives $T = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $E = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, $T^{-1} = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, and

$TET^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = A.$

Furthermore, a different choice of eigenvectors will give a different invertible matrix,

such as the eigenvalue-eigenvector pairs $\left(3, \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)$ and $\left(-1, \begin{bmatrix} -2 \\ 2 \end{bmatrix}\right)$ giving $U = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$,

D as before, and $U^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{-1}{4} & \frac{1}{4} \end{bmatrix}$. Of course; $UDU^{-1} = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{-1}{4} & \frac{1}{4} \end{bmatrix} =$

$= \begin{bmatrix} 6 & 2 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{-1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = A$ is still true.

$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\left(2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, $\left(0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$. Now $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$, and $SDS^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = B.$

$D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\left(i, \begin{bmatrix} i \\ 1 \end{bmatrix}\right)$, $\left(-i, \begin{bmatrix} -i \\ 1 \end{bmatrix}\right)$. Now $S = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $S^{-1} = \begin{bmatrix} \frac{-i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$,

and $SDS^{-1} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} \frac{-i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ i & -i \end{bmatrix}\begin{bmatrix} \frac{-i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = D.$

Example: Finalize diagonalizing the following matrices given these eigenvalue-eigenvector pairs.

$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $\left(1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$, $\left(1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$, $\left(4, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

Then $S = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, and $S^{-1} = \begin{bmatrix} \frac{-1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

$B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$, $\left(1, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix}\right)$, $\left(-\frac{1}{2}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$, $\left(-\frac{1}{6}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right)$.

Then $S = \begin{bmatrix} \frac{2}{3} & -1 & \frac{-1}{2} \\ \frac{2}{3} & 1 & \frac{-1}{2} \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \\ 0 & 0 & \frac{-1}{6} \end{bmatrix}$, and $S^{-1} = \begin{bmatrix} \frac{3}{7} & \frac{3}{7} & \frac{3}{7} \\ \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{-3}{7} & \frac{-3}{7} & \frac{4}{7} \end{bmatrix}$.

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $\left(2, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$, $\left(i, \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}\right)$, $\left(-i, \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}\right)$.

Then $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & -i \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$, and $S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-i}{2} & \frac{1}{2} \\ 0 & \frac{i}{2} & \frac{1}{2} \end{bmatrix}$.

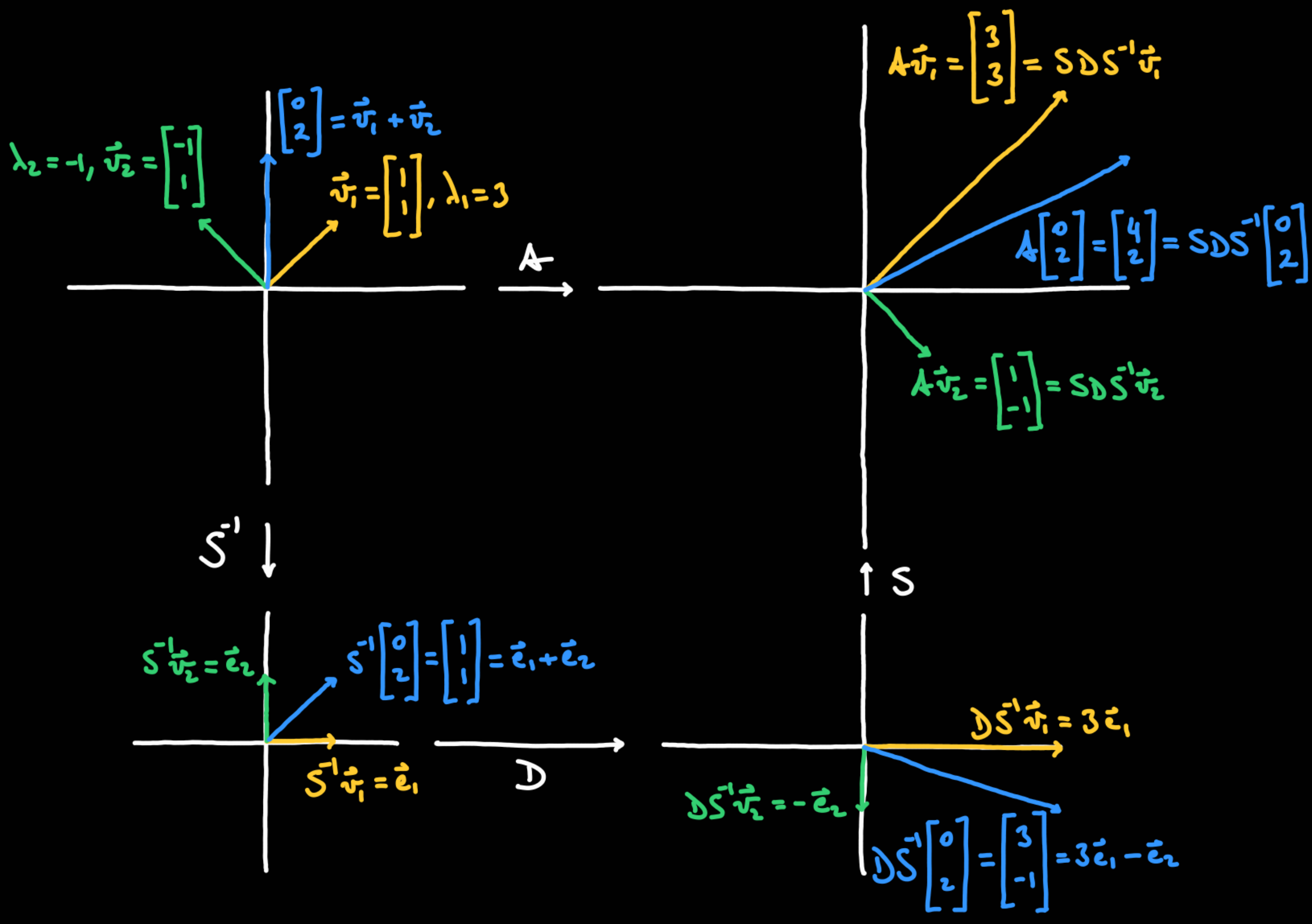
Geometrically, a diagonalizable matrix A has enough eigenvectors to be similar to a diagonal matrix, but ~~not~~ quite identical. If we write any vector in terms of an eigenbasis of A, then we can multiply by A almost as easily as if A was a diagonal matrix, as we have seen before. The equality $A = SDS^{-1}$ is precisely encoding this:

(i) the matrix S changes from an eigenbasis of A to the standard basis,

(ii) the matrix D rescales the standard basis,

(iii) the matrix S^{-1} changes from the standard basis to an eigenbasis of A.

Let's do the very concrete example $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = SDS^{-1}$.



Given a vector $\vec{x}$ in $\mathbb{R}^m$, the multiplication $A\vec{x}$ is best visualized in terms of eigenvectors of A, while the multiplication $SDS^{-1}\vec{x}$ is interpreting $\vec{x}$ as expressed in terms of eigenvectors of A, changing to the standard basis via S^{-1}, rescaling its entries via D, and changing back to the eigenvectors of A via S.

It is useful to have a geometric understanding of the linear transformation to make educated guesses for its eigenvalues, and whether it may be diagonalizable.

(i) Projections are diagonalizable:

$\begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix}$ is projecting onto $2x-y+2z=0$ and scaling everything by 9.

The normal vector is then an eigenvector of eigenvalue zero: $\left(0, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\right)$.

Any vector in the plane $2x-y+2z=0$ will be an eigenvector. To describe the full plane, take two linearly independent vectors: $\left(9, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right), \left(9, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)$.

(ii) Rotations are diagonalizable over the complex numbers:

$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is a rotation of $\frac{\pi}{2}$ on the x_1-x_4-plane and on the x_2-x_3-plane.

Its eigenvalue-eigenvector pairs are the same as for $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, the rotation of $\frac{\pi}{2}$ in two dimensions, just arranged to be in the x_1-x_4-plane and in the

x_2-x_3-plane: $\left(i, \begin{bmatrix} i \\ 0 \\ 0 \\ 1 \end{bmatrix}\right), \left(-i, \begin{bmatrix} -i \\ 0 \\ 0 \\ 1 \end{bmatrix}\right), \left(i, \begin{bmatrix} 0 \\ i \\ 1 \\ 0 \end{bmatrix}\right), \left(-i, \begin{bmatrix} 0 \\ -i \\ 1 \\ 0 \end{bmatrix}\right).$

(iii) Matrices A satisfying $A^n = \mathrm{Id}_n$ are diagonalizable over the complex numbers:

these matrices are "essentially" rotations.

(iv) Matrices A satisfying $A^T = A$ are diagonalizable:

$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$ has eigenvalue-eigenvector pairs: $\left(-1, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right), \left(2+\sqrt{3}, \begin{bmatrix} 1 \\ 1-\sqrt{3} \\ 1 \end{bmatrix}\right), \left(2-\sqrt{3}, \begin{bmatrix} 1 \\ 1+\sqrt{3} \\ 1 \end{bmatrix}\right).$

This is a fundamental result called the Spectral Theorem.

<u>Remark:</u> A useful trick to compute solutions of $A\vec{x} = \vec{0}$ is to eyeball a linear combination of the columns of A that adds up to zero.

(i) The columns of $\begin{bmatrix} -4 & 2 & -4 \\ 2 & -1 & 2 \\ -4 & 2 & -4 \end{bmatrix}$ satisfy $(\pm 1)\cdot c_1 + (\mp 1)\cdot c_3 = \vec{0}$, $(\pm 1)\cdot c_1 + (\mp 2)\cdot c_2 = \vec{0}$,

and $(\pm 2)\cdot c_2 + (\mp 1)\cdot c_3 = \vec{0}$, so $\begin{bmatrix} -4 & 2 & -4 \\ 2 & -1 & 2 \\ -4 & 2 & -4 \end{bmatrix}\begin{bmatrix} \pm 1 \\ 0 \\ \mp 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -4 & 2 & -4 \\ 2 & -1 & 2 \\ -4 & 2 & -4 \end{bmatrix}\begin{bmatrix} \pm 1 \\ \mp 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -4 & 2 & -4 \\ 2 & -1 & 2 \\ -4 & 2 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ \pm 2 \\ \mp 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(ii) The columns of $\begin{bmatrix} i & 0 & 0 & -1 \\ 0 & i & -1 & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & i \end{bmatrix}$ satisfy $(\pm i)\cdot c_1 + (\mp 1)\cdot c_4 = \vec{0}$ and $(\pm i)\cdot c_2 + (\mp 1)\cdot c_3 = \vec{0}$,

so $\begin{bmatrix} i & 0 & 0 & -1 \\ 0 & i & -1 & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & i \end{bmatrix}\begin{bmatrix} \pm i \\ 0 \\ 0 \\ \mp 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} i & 0 & 0 & -1 \\ 0 & i & -1 & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & i \end{bmatrix}\begin{bmatrix} 0 \\ \pm i \\ \mp 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(iii) The columns of $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$ satisfy $(\pm 1)\cdot C_1 + (\mp 1)\cdot C_3 = \vec{0}$, so $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} \pm 1 \\ 0 \\ \mp 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

This could be useful to quickly compute eigenvectors of a matrix, as above:

(i) computes eigenvectors of $\begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix}$ of eigenvalue 9,

(ii) computes eigenvectors of $\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ of eigenvalue i,

(iii) computes eigenvectors of $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$ of eigenvalue -1.

<u>Example:</u> Determine for which values of a, b, d the matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is diagonalizable. For those values of a, b, c diagonalize it.

Since $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is an upper-triangular matrix, its eigenvalues are exactly its diagonal entries, so $\lambda_1 = a$ and $\lambda_2 = d$. The solutions of $\begin{bmatrix} a-\lambda & b \\ 0 & d-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $\lambda = a, d$ depend on whether $a = d$ or $a \neq d$. For $\lambda_1 = a$, if $a = d$ then:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a-a & b \\ 0 & d-a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \text{ has augmented matrix } \left[\begin{array}{cc|c} 0 & b & 0 \\ 0 & 0 & 0 \end{array}\right]$$

whose reduced form depends on whether $b = 0$ or $b \neq 0$. If $b = 0$ then:

$$\left[\begin{array}{cc|c} 0 & b & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ has reduced row echelon form } \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so $y = s$, $x = t$, and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ s \end{bmatrix} = t\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} + s\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. If $b \neq 0$ then:

$$\left[\begin{array}{cc|c} 0 & b & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ has reduced row echelon form } \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so $y=0$, $x=s$, and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ 0 \end{bmatrix} = s\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If $a \neq d$ then:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a-a & b \\ 0 & d-a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \text{ has augmented matrix } \left[\begin{array}{cc|c} 0 & b & 0 \\ 0 & d-a & 0 \end{array}\right]$$

whose reduced form does <u>not</u> depend on whether $b=0$ or $b\neq 0$:

$$\left[\begin{array}{cc|c} 0 & b & 0 \\ 0 & d-a & 0 \end{array}\right] \text{ has reduced row echelon form } \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so $y=0$, $x=s$, and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ 0 \end{bmatrix} = s\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. For $\lambda_2 = d$, if $a=d$ then:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a-d & b \\ 0 & d-d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \text{ has augmented matrix } \left[\begin{array}{cc|c} 0 & b & 0 \\ 0 & 0 & 0 \end{array}\right]$$

whose reduced form depends on whether $b=0$ or $b\neq 0$. As we have seen, if $b=0$

then $\begin{bmatrix} x \\ y \end{bmatrix} = t\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} + s\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and if $b\neq 0$ then $\begin{bmatrix} x \\ y \end{bmatrix} = s\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If $a\neq d$ then:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a-d & b \\ 0 & d-d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \text{ has augmented matrix } \left[\begin{array}{cc|c} a-d & b & 0 \\ 0 & 0 & 0 \end{array}\right]$$

whose reduced form depends on whether $b=0$ or $b\neq 0$. If $b=0$ then:

$$\left[\begin{array}{cc|c} a-d & b & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ has reduced row echelon form } \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so $y=s$, $x=0$, and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ s \end{bmatrix} = s\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. If $b\neq 0$ then:

$$\left[\begin{array}{cc|c} a-d & b & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ has reduced row echelon form } \left[\begin{array}{cc|c} 1 & \frac{b}{a-d} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so $y=s$, $x=\frac{-b}{a-d}\cdot s$, and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-bs}{a-d} \\ s \end{bmatrix} = s\cdot\begin{bmatrix} \frac{-b}{a-d} \\ 1 \end{bmatrix} = s\cdot\begin{bmatrix} b \\ d-a \end{bmatrix}$.

Altogether, we have the following cases with the corresponding eigenvalues and eigenvectors.

$$\begin{cases} a=d & \begin{cases} b=0: & \left(a, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(a, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right). & \text{Diagonal.} \\ b\neq 0: & \left(a, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right). & \underline{\text{Not}} \text{ diagonalizable.} \end{cases} \\ a\neq d & \begin{cases} b=0: & \left(a, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(d, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right). & \text{Diagonal.} \\ b\neq 0: & \left(a, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(d, \begin{bmatrix} b \\ d-a \end{bmatrix}\right). & \text{Diagonalizable.} \end{cases} \end{cases}$$

$$\begin{cases} a=d & \begin{cases} b=0: & \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}. \\ b\neq 0: & \underline{\text{Not}} \text{ diagonalizable.} \end{cases} \\ a\neq d & \begin{cases} b=0: & \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}. \\ b\neq 0: & \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & d-a \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & \frac{-b}{d-a} \\ 0 & \frac{1}{d-a} \end{bmatrix}. \end{cases} \end{cases}$$

6.5.2. Computing high powers of a matrix.

The powers of a diagonalizable matrix are easy to compute. If $A = SDS^{-1}$ with D a diagonal matrix, then $A^n = SD^nS^{-1}$, which only requires taking powers of the non-zero entries of D.

<u>Example:</u> Compute $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}^n$ for all natural numbers n.

We have the same cases as above:

$$(a=d,\ b=0): \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}^n = \begin{bmatrix} a^n & 0 \\ 0 & a^n \end{bmatrix},$$

$$(a=d,\ b\neq 0): \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^n = \begin{bmatrix} a^n & na^{n-1}b \\ 0 & a^n \end{bmatrix},$$

$$(a\neq d,\ b=0): \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}^n = \begin{bmatrix} a^n & 0 \\ 0 & d^n \end{bmatrix},$$

$$(a\neq d,\ b\neq 0): \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}^n = \begin{bmatrix} 1 & b \\ 0 & d-a \end{bmatrix}\begin{bmatrix} a^n & 0 \\ 0 & d^n \end{bmatrix}\begin{bmatrix} 1 & \frac{-b}{d-a} \\ 0 & \frac{1}{d-a} \end{bmatrix} = \begin{bmatrix} a^n & \frac{b(a^n-d^n)}{a-d} \\ 0 & d^n \end{bmatrix}.$$