Math 110AH — Homework 1 28 September 2021

Name and UCLA ID:

Task 1: Read Sections 5 and 6.

Exercise 1: Let a and n be positive integers. Show that if $a^n - 1$ is prime and $n > 1$ then $a = 2$ and n is prime. Suppose that $2^n + 1$ is prime, what can you say about n?

Exercise 2: Let x, y, z, n, a, and b be integers. Show that:

1. If $x|y$ and $x|z$ then $x|ay + bz$.
2. If $x|y$ then $x|yn$.
3. If $x|y$ and $y \neq 0$ then $|y| \geq |x| \geq x$.
4. If $xy = 0$ then $x = 0$ or $y = 0$.
5. If $xa = xb$ then $x = 0$ or $a = b$.

Exercise 3: Let a, b, and n be positive integers with $n > 1$. Determine when $n^{\frac{a}{b}}$ is rational, and prove your result. You can use the Fundamental Theorem of Arithmetic.

Exercise 4: Prove that the cartesian product of finitely many countable sets is countable.

Exercise 5: Prove that any two (finite) line segments have the same cardinality.

Exercise 6: Let $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$ (or any field). Let $F[t]$ be the set of polynomials with coefficients in F, with the usual addition and multiplication. State and prove the analog of the Division Algorithm for Integers (use your knowledge of such division. Use degrees of polynomials as a substitute for the second statement in the Division Algorithm). What can you do if you take polynomials with coefficients in $\mathbb{Z}$?

Exercise 7: Prove that the number of subsets of a set with n elements is 2^n.

Exercise 8: The first nine Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34. What is the n-th Fibonacci number F_n? Show that $F_n < 2^n$.

Exercise 9: Euclid's proof of the infinitude of primes shows that if p_n is the n-th prime then $p_{n+1} \leq p_n^n + 1$. Show that $p_{n+1} \leq 2^{2^{n+1}}$. Using this, show that if $x \geq 2$ then $\pi(x) \geq \log(\log(x))$ where $\pi(x)$ is the number of primes less than x (this is a bad estimate).

Exercise 10: When Gauss was ten years old he recognized that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ (what he did was a bit harder). What is a formula for the sum of the first n cubes? Prove your result.