

NAME AND UCLA ID:

Task 1: Read Sections 12 and 13.**Exercise 1:** Let $\varphi : G \rightarrow H$ a group homomorphism. Prove that

1. $\ker(\varphi) \subseteq G$ and $\text{im}(\varphi) \subseteq H$ are subgroups.
2. φ is an isomorphism if and only if φ is bijective.

Exercise 2: Besides itself, to what other group is $\text{ST}_2(\mathbb{R})$ isomorphic? Prove your claim.**Exercise 3:** Let $p \in \mathbb{Z}$, $p > 1$. Wilson's Theorem states that p is prime if and only if $(p-1)! \equiv -1 \pmod{p}$. Prove Wilson's Theorem. Hint: Let $1 \leq j \leq p-1$, when do we have $j^2 \equiv 1 \pmod{p}$?**Exercise 4:** Let G be a group, H and K subgroups of G . Prove the following:

1. If $H \subseteq K \subseteq G$ and H has finite index in G , then $[G : H] = [G : K][K : H]$.
2. Let $HK = \{hk | h \in H, k \in K\}$. Then $H/(H \cap K)$ is a subset of $G/(H \cap K)$ and $(HK)/K$ is a subset of G/K . Show that $f : H/(H \cap K) \rightarrow (HK)/K$ given by $f(h(H \cap K)) = hK$ for all $h \in H$ is a well-defined bijection.
3. If both H and K have finite index in G , then $H \cap K$ has finite index in G .

Exercise 5: Prove that $\text{Aut}(G)$ is a group and $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$. For G a cyclic group, determine $\text{Aut}(G)$ and $\text{Inn}(G)$ up to isomorphism as groups that we know.**Exercise 6:** Show that a subgroup $H \subseteq G$ is normal if and only if $gH = Hg$ for all $g \in G$. If H is not normal, is it true that for each $g \in G$ there is an $a \in G$ with $gH = Ha$?**Exercise 7:** Find all subgroups of S_3 and determine which ones are normal.**Exercise 8:** Let G be a group of order p^n for $p \in \mathbb{Z}^+$ prime and $n \in \mathbb{Z}^+$. Prove that there exists an element of order p in G .**Exercise 9:** Prove that a group of order 30 can have at most 7 subgroups of order 5.