MATH 110AH — HOMEWORK 5 1 NOVEMBER 2021

NAME AND UCLA ID:

Task 1: Read Sections 14, 15, 16, 17, and 18.

Exercise 1: Let $H \subsetneq G$ be a subgroup of a finite group G. Suppose that $|G|$ does not divide $[G : H]!$. Prove that G contains a proper normal subgroup N, and that N is a subgroup of H. In particular, G is not simple. This is a useful counting result.

Exercise 2: Let $f : A \to B$ be a map of sets. If $D \subseteq B$ is a subset we define the preimage of D in A as the set $f^{-1}(D) = \{a \in A | f(a) \in D\}$. Let $C \subseteq A$ and $D \subseteq B$, prove the following properties of the preimages:

1. $C \subseteq f^{-1}(f(C))$ with equality if f is injective.
2. $f(f^{-1}(D)) \subseteq D$ with equality if f is surjective.

Exercise 3: Let $K \trianglelefteq G$, $L \leq G/K$, and $\varphi : G \to G/K$ given by $\varphi(g) = gK$. Show that:

1. There exists a subgroup H of G containing K with $L = H/K$.
2. If $H \leq G$ containing K with $L = H/K$, then $L \trianglelefteq G/K$ if and only if $H \trianglelefteq G$.
3. Let H_1 and H_2 be two subgroups of G containing K. If $H_1/K = H_2/K$ then $H_1 = H_2$.
4. If G is a finite subgroup and $H \leq G$ containing K with $L = H/K$, then $[G : H] = [G/K : H/K] = [G/K : L]$ and $|H| = |K||L|$.

This is an alternate form of the Correspondence Principle.

Exercise 4: Let G be a group. Show that:

1. $Z(G)$ is a subgroup of G.
2. $Z(G)$ is normal in G.
3. G is abelian if and only if $Z(G) = G$ (you cannot use this to prove the above).
4. Let $a \in G$, the centralizer of a in G is defined as $Z_G(a) = \{x \in G | xa = ax\}$. Then $Z_G(a)$ is a subgroup of G. Moreover $Z(G) = \bigcap_{a \in G} Z_G(a)$.
5. Let $a \in G$, the conjugacy class of a in G is defined as $C(a) = \{xax^{-1} | x \in G\}$. Then $a \in Z(G)$ if and only if $C(a) = \{a\}$ if and only if $|C(a)| = 1$ if and only if $G = Z_G(a)$.