Math 110AH — Homework 8 12 November 2021

Name and UCLA ID:

Task 1: Read Sections 1 through 24.

Exercise 1: Let G be a group such that for three consecutive integers $i \in \mathbb{Z}^+$ we have $(ab)^i = a^i b^i$ for all $a, b \in G$. Prove that G is abelian.

Exercise 2: Let $Q = \langle -1, i, j, k | (-1)^2 = 1, i^2 = j^2 = -1, k = ij = -ji \rangle$. Show that Q is a group of eight elements and has the following properties:

1. All subgroups of Q are normal, but Q is not abelian.
2. The group Q contains a unique subgroup of order two.
3. There exist no proper subgroups H and K of Q, at least one of them being normal, such that $Q = HK$ and $H \cap K = 1$.
4. There does not exist a group monomorphism from Q into S_7.

Exercise 3: A *commutator* of a group G is an element of the form $xyx^{-1}y^{-1}$ for some $x, y \in G$. Denote by G' the subgroup of G generated by all commutators (namely every element of G' is a finite product of commutators and inverses of commutators), we say that G' is the *commutator subgroup* or *derived subgroup* of G (it is also denoted by $[G, G]$). Show that the following are true.

1. $G' \trianglelefteq G$.
2. G/G' is abelian.
3. If $N \trianglelefteq G$ and G/N is abelian, then $G' \subseteq N$.
4. If $H \leq G$ is a subgroup with $G' \subseteq H$, then $H \trianglelefteq G$.

Exercise 4: Recall that $\text{Aut}(G) = \{f : G \to G | f \text{ isomorphism}\}$, and that a subgroup H of G is called *characteristic* in G, denoted $H \lhd\lhd G$, if for every $f \in \text{Aut}(G)$ then $f|_H \in \text{Aut}(H)$. Show that the following are true.

1. If $K \lhd\lhd H$ and $H \lhd G$ then $K \lhd G$.
2. $Z(G) \lhd\lhd G$.
3. $G' \lhd\lhd G$.
4. Define $G^{(n)}$ inductively via $G^{(1)} = G'$ and $G^{(n+1)} = \left(G^{(n)}\right)'$. Show that $G^{(n)} \lhd\lhd G$ for all $n \in \mathbb{Z}^+$.

Exercise 5: A group G is called *solvable* if there exist distinct subgroups N_i with $i = 1, \dots, r$ for some $r \in \mathbb{Z}^+$ such that $N_1 = 1$, $N_r = G$, $N_i \triangleleft N_{i+1}$ for $i = 2, \dots, r-1$, and N_{i+1}/N_i is abelian for $i = 2, \dots, r-1$. **Using the isomorphism theorems**, prove the following.

1. A subgroup of a solvable group is solvable.
2. The homomorphic image of a solvable group is solvable.
3. If $N \triangleleft G$, N is solvable, and G/N is solvable, then G is solvable.

Exercise 6: Let $p, q, r \in \mathbb{Z}^+$ be prime. Show that the following are true (you may use the Sylow Theorems without proving them. Unless you prove it, you may not use that a group of order $p^n q$ is solvable for all $n \in \mathbb{Z}^+$).

1. If G is a p-group, then G is solvable.
2. If $|G| \in \{pq, p^2q, p^2q^2, pqr\}$, then G is solvable.

Exercise 7: Let G be a finite group, $H \leq G$ a p-group for $p \in \mathbb{Z}^+$ prime, and suppose that p divides $[G : H]$. show that p divides $[N_G(H) : H]$ and that $H \leq N_G(H)$.

Exercise 8: Let G be a finite group and H a proper subgroup. Show that $G \neq \bigcup_{g \in G} gHg^{-1}$.

Exercise 9: Let G be a finite group, and S be a finite a G-set. For $x \in G$ let $F_S(x) = \{s \in S | x * s = s\}$ and denote by N the number of the action. Prove that:

$$N = \frac{1}{|G|} \sum_{x \in G} |F_S(x)|.$$

Exercise 10: Let $p \in \mathbb{Z}^+$ be an odd prime. Classify, up to isomorphism, all groups G of order $2p$.