Definition: Let $p \in \mathbb{Z}^+$ be a prime and $n \in \mathbb{Z}^+$. We denote the image of the group homomorphism $\varphi : \frac{\mathbb{Z}}{p^n\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}}{p^n\mathbb{Z}}$, $\bar{x} \longmapsto \overline{px}$ by $p \cdot \left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right)$.

Proposition: We have $p \cdot \left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right) \cong \frac{\mathbb{Z}}{p^{n-1}\mathbb{Z}}$.

Proof: The kernel of φ is $\ker(\varphi) = \{\overline{k \cdot p^{n-1}} \mid k \in \{0, \dots, p-1\}\}$.

$\supseteq$) Clearly $\varphi(\overline{k \cdot p^{n-1}}) = \overline{p \cdot k \cdot p^{n-1}} = \overline{k \cdot p^n} = \bar{0}$.

$\subseteq$) Let $\bar{x} \in \ker(\varphi)$, then $\bar{0} = \varphi(\bar{x}) = \overline{p \cdot x}$ so p^n divides $p \cdot x$. Pick x a representative smaller than p^n, then p^{n-1} divides x, so $x = k \cdot p^{n-1}$ for some $k \in \{0, 1, \dots, p-1\}$.

Moreover, φ is surjective onto its image, so by the First Isomorphism Theorem:

$$\bar{\varphi} : \frac{\left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right)}{\ker(\varphi)} \xrightarrow{\cong} \operatorname{im}(\varphi) = p \cdot \left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right) \text{ is an isomorphism.}$$

Since $\frac{\mathbb{Z}}{p^n\mathbb{Z}}$ is cyclic, and the quotient of a cyclic group is cyclic, we have that $\frac{\left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right)}{\ker(\varphi)}$ is a cyclic group of order $\left| \frac{\left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right)}{\ker(\varphi)} \right| = \frac{\left|\left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right)\right|}{|\ker(\varphi)|} = \frac{p^n}{p} = p^{n-1}$ by Lagrange's Theorem. By the Classification of Cyclic groups we have that

$$\frac{\left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right)}{\ker(\varphi)} \cong \frac{\mathbb{Z}}{p^{n-1}\mathbb{Z}}, \text{ so } \frac{\mathbb{Z}}{p^{n-1}\mathbb{Z}} \cong \frac{\left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right)}{\ker(\varphi)} \cong \operatorname{im}(\varphi) = p \cdot \left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right). \qquad \square$$

Recall: If $G = \langle x \rangle$ and $H \trianglelefteq G$, then $\frac{G}{H} = \langle xH \rangle$.