(1) Let R be a ring. An element $a \in R$ is called nilpotent whenever $a^n = 0$ for some $n \in \mathbb{Z}^+$.

Suppose that R is commutative and $a, b \in R$ are nilpotent. Prove that $a+b$ is nilpotent.

Show that this is <u>not</u> necessarily true if R is <u>not</u> commutative by giving a counterexample.

For R commutative and a, b nilpotent we have $a^n = 0$, $b^m = 0$ for some $n, m \in \mathbb{Z}^+$. Then

$(a+b)^{n+m} = \sum_{i,j} c_{ij} a^i b^j$ for some $c_{ij} \in \mathbb{Z}^+$ and either $i \geq n$ or $j \geq m$, so $(a+b)^{n+m} = 0$.

For $R = M_2(\mathbb{Z})$ consider $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $a^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = b^2$ but

$a + b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $(a+b)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $(a+b)^n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for n odd and $(a+b)^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

for n odd. Thus $(a+b)^n \neq 0$ for all $n \in \mathbb{Z}^+$, so $a+b$ is <u>not</u> nilpotent.

(2) Let $f: G \longrightarrow H$ be a group homomorphism, H abelian, $N \leq G$ with $\ker(f) \subseteq N$. Prove that

N is normal in G.

By the First Isomophism Theorem $G/\ker(f) \cong f(H)$, and since $f(H) \leq H$ is a subgroup of a

commutative group, it is commutative. By the Correspondence Theorem, a subgroup $N \leq G$

with $\ker(f) \subseteq N$ corresponds to a subgroup $N/\ker(f) \leq G/\ker(f)$. Since a subgroup of an abelian

group is normal, $N/\ker(f) \trianglelefteq G/\ker(f)$. Again by the Correspondence Theorem, a normal

subgroup $N/\ker(f) \trianglelefteq G/\ker(f)$ corresponds to a normal subgroup $N \trianglelefteq G$ containing $\ker(f)$.

(3)

③ Let G be a finite group and $H \leq G$ of order n. If H is the only subgroup of G of order n, then H is normal in G.

Prove that $f_g : H \longrightarrow gHg^{-1}$, $h \longmapsto ghg^{-1}$ is an isomorphism of subgroups of G for all $g \in G$. Then $|H| = |gHg^{-1}|$, so $H = gHg^{-1}$ and $H \trianglelefteq G$.

④ Let $N_1 \trianglelefteq G_1$, $N_2 \trianglelefteq G_2$. Then $N_1 \times N_2 \trianglelefteq G_1 \times G_2$ and $\frac{G_1 \times G_2}{N_1 \times N_2} \cong \frac{G_1}{N_1} \times \frac{G_2}{N_2}$.

Consider $f : G_1 \times G_2 \longrightarrow \frac{G_1}{N_1} \times \frac{G_2}{N_2}$, $(g_1, g_2) \longmapsto (g_1N_1, g_2N_2)$, this is a group homomorphism. Moreover

$N_1 \times N_2 = \ker(f)$ and $\ker(f) \trianglelefteq G_1 \times G_2$, f is surjective, so we are done by the First Isomorphism Theorem.

⑤ Let G be a group, H a normal cyclic subgroup of G. Then every subgroup of H is normal in G.

Let $H = \langle h \rangle$ for some $h \in H$. Since $H \trianglelefteq G$, fixing $g \in G$ gives an $n \in \mathbb{Z}^+$ with $ghg^{-1} = h^n$.

Let $K \leq H$, we have $H = \langle h^m \rangle$ for some $m \in \mathbb{Z}^+$. Now for a fixed $g \in G$ we have:

$$gh^mg^{-1} = \overbrace{ghg^{-1}hg \cdots ghg^{-1}}^{m} = (ghg^{-1})^m = (h^n)^m = (h^m)^n \quad \text{so} \quad gKg^{-1} \leq K \quad \text{so} \quad K \trianglelefteq G.$$

⑥ Show that A_4 is <u>not</u> simple.

We have that $K = \{e, (12)(34), (13)(24), (14)(23)\}$ is normal in A_4. This is called the Klein four group.

⑦ Prove that the quaternion group Q_8 is <u>not</u> isomorphic to the dihedral group D_4.

D_4 has two elements of order two: f, r^2.

Q_8 has one element of order two: -1.