Math 110AH
Algebra (Honors)

Practice Problems for November 29, 2021

Problem 1.
Let $n \in \mathbb{Z}^+$, $n \leq 2$. Prove that A_n is the only subgroup of S_n of index two.

Solution: For $n = 2$ then $S_2 \cong \mathbb{Z}/2\mathbb{Z}$ and $A_2 = \{0\}$ is the only subgroup of index two. For $n > 2$, then S_n contains a three cycle α. Let $T < S_n$ be a subgroup of index two and suppose that $\alpha \notin T$, so $S_n/T = \{T, \alpha T\}$. If $\alpha^{-1} \in T$ then all the powers of α^{-1} are also in T, a contradiction. Hence $\alpha^{-1} \notin T$ and both α and α^{-1} are in the coset αT. But then acting by left multiplication on G/T we have $\alpha T = T$, a contradiction. Hence $\alpha \in T$ for any three cycle $\alpha \in S_n$. Since T contains all three cycles, it must contain A_n, and since both T and A_n have index two, they must be equal.

Problem 2.
Let $n \in \mathbb{Z}^+$, $n \leq 3$. Find all normal subgroups of D_n.

Solution: Let $D_n = \langle r, f | f^2 = r^n = frfr = e \rangle$. Now $\langle r \rangle$ is normal in D_n, so all its subgroups are also normal. Noticing that $r^i f$ is conjugate to $a^{i+2} f$, we either have one conjugacy class containing f, or we have the two conjugacy classes $\{a^{2i} f\}_{i \in \mathbb{Z}}$ and $\{a^{2i+1} f\}_{i \in \mathbb{Z}}$. Thus for n odd the normal subgroups are of the form $\langle r^i \rangle$ for i dividing n, and for n even the normal subgroups are of the form $\langle r^i \rangle$ for i dividing n, $\langle r^2, f \rangle$, and $\langle r^2, rf \rangle$.

Problem 3.
Let $n \in \mathbb{Z}^+$, $n \leq 3$. Find the center of D_n.

Solution: For n even $Z(D_n) = \langle r^{n/2} \rangle \cong \mathbb{Z}/2\mathbb{Z}$, and for n odd $Z(D_n) = \langle e \rangle$. To see this, note that an element of the form r^i is central if and only if $r^i = r^{-1}$ so $i = n/2$ so n must be even, and if an element of the form $r^i f$ is central then $n = 2$.

Problem 4.
Let S be an infinite set, define $H = \{\sigma \in \Sigma(S) | \sigma(x) = x \text{ for all but finitely many } x\}$. Prove that H is a subgroup of $\Sigma(S)$.

Solution: Let $\sigma, \tau \in H$, we have that $\sigma(x) \neq x$ if and only if $x \in X$ for some finite set $X \subset S$, and $\tau(x) \neq x$ if and only if $x \in Y$ for some finite set $Y \subset S$. Thus $\sigma(\tau(x)) \neq x$ at most for $x \in X \cup Y$, which is a finite set.

Problem 5.

Let G and H be finite cyclic groups. Prove that $G \times H$ is cyclic if and only if $\gcd(|G|, |H|) = 1$.

Solution: Let $G = \langle g \rangle$, $|G| = m$, $H = \langle h \rangle$, $H = n$. ($\Rightarrow$) Let $\gcd(|G|, |H|) = k$ and $G \times H$ be cyclic. Then $\langle (g^{m/k}, 0) \rangle$ and $\langle (0, h^{n/k}) \rangle$ are two distinct subgroups of $G \times H$ of order k. Since cyclic groups have unique subgroups of each order, we must have $g^{m/k} = 0 = h^{n/k}$, which is only possible if $k = 1$. ($\Leftarrow$) Let $\gcd(|G|, |H|) = 1$, then $G \times H$ is an abelian group with elements $\langle (g, 0) \rangle$ and $\langle (0, h) \rangle$ of orders m and n respectively. Since $\gcd(|G|, |H|) = 1$ then $\langle (g, h) \rangle$ has order mn, and since $|G \times H| = |G||H| = mn$ then we must have $G \times H = \langle (g, h) \rangle$.

Problem 6.

Let $n \in \mathbb{Z}^+$, $n \leq 3$. Find the center of S_n.

Solution: Let $\sigma \in S_n$, $\sigma \neq e$, so $\sigma(i) = j$ for some $i \neq j$. Pick $k \neq i, j$, then $\sigma(ik) \neq (ik)\sigma$, so $Z(S_n) = \{e\}$.

Problem 7.

Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| a, b, c \in \mathbb{R},\ ad \neq 0 \right\}$. Determine whether H is a normal subgroup of $\mathrm{GL}_2(\mathbb{R})$ or not.

Solution: For $x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ we have $x \in H$, $y \notin H$, and $yxy^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \notin H$. Hence H is not normal in $\mathrm{GL}_2(\mathbb{R})$.