# Math 110AH
# Algebra (Honors)

## Practice Problems for December 1, 2021

**Problem 1.**
Find all the elements of $M_2(\mathbb{Z})$ that have multiplicative inverses.

> **Solution:** Let $A, B \in M_2(\mathbb{Z})$ with $AB = 1$. Then $1 = \det(1) = \det(AB) = \det(A)\det(B)$ so $\det(A) = \pm 1$. The converse is also true.

**Problem 2.**
Let $G$ be a cyclic group of order $n \in \mathbb{Z}^+$ divisible by $k \in \mathbb{Z}^+$. Prove that $G$ has exactly one subgroup of order $k$.

> **Solution:** Let $G = \langle g \rangle$, the unique subgroup of order $k$ is the subgroup $\langle g^{n/k} \rangle$.

**Problem 3.**
Let $G$ be a finite group. Prove that the following are equivalent.

1. $G$ has prime order.
2. $G$ is not trivial and $G$ has no proper subgroups.
3. $G \cong \mathbb{Z}/p\mathbb{Z}$ for some prime $p \in \mathbb{Z}^+$.

> **Solution:** (1. $\Rightarrow$ 2.) If $G$ has prime order then $G$ is not trivial. Also, $G$ has no proper subgroups by Lagrange's Theorem.
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> (2. $\Rightarrow$ 3.) If $G$ is not trivial and has no proper subgroups, then there is an element $g \in G$ such that $g \neq e$, and then the subgroup $\langle g \rangle$ must be the whole $G$. By the classification of cyclic groups, we have $G \cong \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{Z}^+$. If $n$ is composite then there exists a proper subgroup of $G$, so $n$ must be prime.
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> (3. $\Rightarrow$ 1.) If $G \cong \mathbb{Z}/p\mathbb{Z}$ for some prime $p \in \mathbb{Z}^+$, then $G$ has order the prime $p$.

**Problem 4.**
Let $p \in \mathbb{Z}^+$ be prime, $n \in \mathbb{Z}^+$. Define $p(\mathbb{Z}/p^n\mathbb{Z})$ and prove that $p(\mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{Z}/p^{n-1}\mathbb{Z}$.

> **Solution:** Let $\varphi : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ be the multiplication by $p$. We define $\operatorname{im}(\varphi) = p(\mathbb{Z}/p^n\mathbb{Z})$, we have $\ker(\varphi) = \{\overline{kp^{n-1}} | k \in \{0, \dots, p-1\}\}$. By the First Isomorphism Theorem $(\mathbb{Z}/p^n\mathbb{Z})/\ker(\varphi) \cong \operatorname{im}(\varphi) = p(\mathbb{Z}/p^n\mathbb{Z})$, and using Lagrange's Theorem and the classification of cyclic groups we have that $(\mathbb{Z}/p^n\mathbb{Z})/\ker(\varphi) \cong \mathbb{Z}/p^{n-1}\mathbb{Z}$.

**Problem 5.**

Let $G$ be a group, $H$ a finitely generated normal subgroup such that $G/H$ is finitely generated. Prove that $G$ is finitely generated.

**Solution:** Let $X \subseteq H$ be the finite set that generates $H$, and $Y' \subseteq G/H$ be the finite set that generates $G/H$. Pick $Y \subseteq G$ the representatives of the cosets of $G/H$ such that $Y' = \{yH | y \in Y\}$. Then $H = \langle x_1, \ldots, x_n \rangle$, so every $h \in H$ can be written as $h = x_1^{r_1} \cdots x_s^{r_s}$. Also $G/H = \langle y_1 H, \ldots, y_m H \rangle$, and every $g \in G$ belongs to exactly one coset. Then if $g \in y_1 H \cdots y_m H$ we have $g = \Pi_{i=1}^m y_i h_i = \Pi_{i=1}^m y_i x_1^{k_1^i} \cdots x_n^{k_n^i}$ so $g \in \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$. Since $X, Y \subseteq G$ we have $G = \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$.

**Problem 6.**

Let $n \in \mathbb{Z}^+$ and $\sigma, \tau \in S_n$. Prove that if $\sigma$ is even then $\tau\sigma\tau^{-1}$ is even. Prove that if $\sigma$ is odd then $\tau\sigma\tau$ is odd.

**Solution:** Note that $\tau$ and $\tau^{-1}$ have the same parity. Hence if $\tau$ decomposes into $n$ transpositions, then $\tau$ and $\tau^{-1}$ contribute with $2n$ transpositions to the decomposition of $\tau\sigma\tau^{-1}$.

**Problem 7.**

Let $n \in \mathbb{Z}^+$. Prove that $S_n$ is generated by $(12)$ and $(1, 2, \cdots, n-1, n)$. Prove that $S_n$ is generated by $(12)$ and $(2, 3, \cdots, n-1, n)$.

**Solution:** We have $(1, 2 \cdots n-1, n)^{i-1}(12)(1, 2 \cdots n-1, n)^{-i+1} = (i, i+1)$, which generates $S_n$. We also have $(2, 3 \cdots n-1, n)^{i-1}(12)(2, 3 \cdots n-1, n)^{-i+1} = (1, i+1)$, which generates $S_n$.