Def.: G acting on S, $s \in S$ the <u>stabilizer</u> subgroup is:

$$G_s = \{x \in G \mid x * s = s\}.$$

Proposition: Let S be a G-set, fix $s \in S$, then $f_s : G/G_s \longrightarrow G * s$, $xG_s \longmapsto x * s$ is a bijection. In particular if $[G : G_s]$ is finite, then $|G * s| = [G : G_s]$, and $|G * s|$ divides $|G|$.

Proof: This f_s is well defined:

$$xG_s = yG_s \Leftrightarrow y^{-1}x \in G_s \Leftrightarrow (y^{-1}x) * s = s \Leftrightarrow y^{-1} * (x * s) = s$$

$$\Leftrightarrow y * (y^{-1} * (x * s)) = y * s \Leftrightarrow \underbrace{(yy^{-1})}_{e} * (x * s) = y * s$$

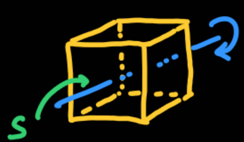
$$\Leftrightarrow x * s = y * s \Leftrightarrow f_s(x) = f_s(y).$$

So f_s is injective (because $\Leftarrow$). Also f_s is surjective since for any $x * s$ we always have $f_s(xG_s) = x * s$. $\square$.

Example: S = faces of a cube  G = group of rotations of the cube.

Given any two faces s_1, s_2, there is an element $g \in G$ taking one to the other. So there is exactly one orbit under this action. (We say that G acts <u>transitively</u> on S)

Fix s a face, the isotropy / stabilizer subgroup of s is the cyclic group of four elements.



we have rotations by $\frac{\pi}{2}$.

By the Proposition: $|G*s| = [G:G_s] = \frac{|G|}{|G_s|}$ so: $|G| = |G_s| \cdot |G*s| = 4 \cdot 6 = 24$.

In general, let S be the regular solid with n-faces, each of them has k edges/vertices, consider G the group of rotations of the faces of S. Then G acts transitively on S,

$|G*s| = n$, $|G_s| = k$, so $|G| = nk$.

Remark that there are only five regular solids (n,k): tetrahedron $(4,3)$, cube $(6,4)$, octahedron $(8,3)$, dodecahedron $(12,5)$, icosahedron $(20,3)$.

Regular solid: solid with n-faces, each face is a regular k-gon.

<u>Definition:</u> S a G-set, fix $s \in S$. We say that s is a <u>fixed point</u> if $G*s = \{s\}$. We denote the set of fixed points of S under the action of G by:

$$F_G(S) = \{s \in S \mid |G*s| = 1\}.$$

<u>Lemma:</u> S a G-set, $s \in S$. The following are equivalent:

(i) $s \in F_G(S)$.

(ii) $G_s = G$.

(iii) $G*s = \{s\}$.

<u>Notation:</u> For $\mathcal{O}$ a system of representatives of the G-action on S, we denote $\mathcal{O}^* = \mathcal{O} \setminus F_G(S)$.

For $s \in \mathcal{O}^*$ then $[G:G_s] = |G*s| > 1$, so $G_s \subsetneq G$.

Theorem: (Orbit decomposition theorem) Let S be a G-set, then:

$$S = F_G(S) \vee \bigvee_{s \in \mho^*} G*s \qquad\qquad S = \bigvee_{\mho} G*s$$

In particular if S is finite then:

$$|S| = |F_G(S)| + \sum_{s \in \mho^*} |G*s| = |F_G(S)| + \sum_{s \in \mho^*} [G : G_s].$$