Lemma: Let G be a finite group, P a Sylow p-subgroup, $H < G$ a p-group. If $H < N_G(P)$ then $H < P$, and if H is a Sylow p-subgroup then $H = P$.

Notation: For G group, let $S = \mathcal{P}(G)$, consider the action
$$\begin{aligned} * : G \times S &\longrightarrow S \\ (x, A) &\mapsto xAx^{-1} \end{aligned}$$
We set:

$C(A) := G * A = \{xAx^{-1} \mid x \in G\}$, now for $H < G$ we can restrict this action to:

$$*|_{G \times C(H)} : G \times C(H) \longrightarrow C(H),$$
making $C(H)$ into a G-set.

Lemma: Let G be a finite group, P a Sylow p-subgroup, $H < G$ a p-group. Then:

1. $C(P)$ consists of Sylow p-subgroups.

2. The conjugacy class $C(P)$ is an H-set by conjugation.

3. If T is a fixed point under the H-action, namely:

   $T \in F_H(C(P)) = \{W \in C(P) \mid xWx^{-1} = W \text{ for all } x \in H\}$, then $H \subseteq T$.

4. If H is a Sylow p-subgroup then under the H-action, H is the only possible fixed point:

   $F_H(C(P)) \subseteq \{H\}$.

Theorem: (Second Sylow Theorem) Let G a finite group, $p \in \mathbb{Z}^+$ a prime dividing $|G|$. Then all Sylow p-subgroups are conjugate: $P \in Syl_p(G)$ then $C(P) = Syl_p(G)$.

Theorem: (Third Sylow Theorem) Let G a finite group, $p \in \mathbb{Z}^+$ a prime dividing $|G|$. Let P be a Sylow p-subgroup, then:

(i) $[G : N_G(P)] = |Syl_p(G)|$.

(ii) $|Syl_p(G)|$ divides $|G|$

(iii) $|Syl_p(G)| \equiv 1 \mod p$

(iv) $|Syl_p(G)|$ divides $[G:P]$, namely if $|G| = p^n m$ with $p \nmid m$ then $|Syl_p(G)| \mid m$.

<u>Theorem:</u> (Fourth Sylow Theorem) Let G a finite group, $p \in \mathbb{Z}^+$ a prime dividing $|G|$. Let H be a p-subgroup of G. Then H lies in some Sylow p-subgroup of G.

<u>Proof:</u> (Second, Third, and Fourth Sylow Theorems) Let H be a subgroup of G that is a p-group, and P is a Sylow p-subgroup of G. By the Lemma above, $C(P) \subseteq Syl_p(G)$ is an H-set under conjugation. By the Orbit Decomposition Theorem:

$$|C(P)| = |F_H(C(P))| + \sum_{T \in \mathcal{O}^*} [H : H_T].$$

If $T \in \mathcal{O}^*$ then $H_T \subsetneq H$ so p divides $[H : H_T]$, so $|C(P)| \equiv |F_H(C(P))| \mod p$.

If $H = P$, by the Lemma above we have $F_P(C(P)) \subseteq \{P\}$. Since $xPx^{-1} = P$ for all $x \in P$ we have $F_P(C(P)) = \{P\}$. Now by the above:

$$|C(P)| \equiv |F_P(C(P))| \equiv 1 \mod p.$$

If H is a Sylow p-subgroup of G, <u>not</u> necessarily P, then:

$|F_H(C(P))| \equiv |C(P)| \equiv 1 \mod p$. Hence $F_H(C(P))$ is <u>not</u> empty, so by the Lemma above, $F_H(C(P)) = \{H\}$. In particular $H \in F_H(C(P)) \subseteq C(P)$. Since H is any Sylow p-subgroup, then $Syl_p(G) \subseteq C(P)$. So $Syl_p(G) = C(P)$. This proves the Second Sylow Theorem. Hence: $|Syl_p(G)| \equiv |C(P)| \equiv 1 \mod p$, also $|Syl_p(G)| = |C(P)| = [G : N_G(P)]$, and $|G| = [G : N_G(P)] \cdot |N_G(P)| = |Syl_p(G)| \cdot |N_G(P)|$ so $|Syl_p(G)|$ divides $|G|$. Moreover since $P \leq N_G(P)$ we have: $[G:P] = [G : N_G(P)][N_G(P) : P] = |Syl_p(G)| \cdot [N_G(P) : P]$ so $|Syl_p(G)|$ divides $[G:P]$, proving the Third Sylow Theorem.

If H is a p-subgroup of G, <u>not</u> necessarily a Sylow p-subgroup, we still have:

$$|F_H(C(P))| \equiv |C(P)| \equiv 1 \mod p.$$

Thus there exists $T \in F_H(C(P)) \subseteq C(P) = Syl_p(G)$, so by the Lemma above $H \subseteq T$, and T is a Sylow p-subgroup. So H is contained in a Sylow p-subgroup, proving the Fourth Sylow Theorem. $\square$.