Normal subgroup that is not abelian:

$A_n \trianglelefteq S_n$ because A_n is the kernel of a group homomorphism.

However, in general A_n is not abelian: for $n=5$, we have $(12)(23) \in A_n$, also $(24)(45) \in A_n$. Now:

$$(12)(23)(24)(45) = (12453) \quad \text{but} \quad (24)(45)(12)(23) = (14523), \text{ they are different.}$$

Symmetries: $\Sigma(S) = \{ f: S \to S \mid f \text{ bijection of sets} \}$.

$\mathrm{Aut}(G) = \{ f: G \to G \mid f \text{ automorphism} \}$.

Normality: G/H for $H \leq G$ is a group if and only if $H \trianglelefteq G$.

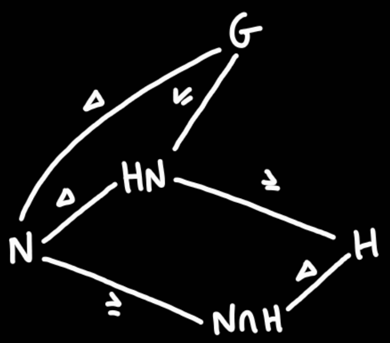
Theorem: G group, $H \leq G$, $N \trianglelefteq G$. Then:

(i) $H \cap N \trianglelefteq H$

(ii) $HN = NH \leq G$.

(iii) $N \trianglelefteq HN$.

(iv) $H/_{H\cap N} \cong HN/_N$



Characteristic: Normal subgroup: $gNg^{-1} = N$ for all $g \in G$. Equivalently if $\sigma \in \mathrm{Inn}(G)$, then

$\sigma|_N \in \mathrm{Aut}(N)$: $\sigma|_N$ fixes N since $\sigma|_N(N) = N$. (recall $\theta_g(N) = gNg^{-1}$).

Characteristic: if $\sigma \in \mathrm{Aut}(G)$ then $\sigma|_N \in \mathrm{Aut}(N)$.

We can think of characteristic subgroups as independent objects inside G that do not interact with anything else: $G = (G \setminus N) \cup N$

$G = \mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, $H = \mathbb{Z}/p\mathbb{Z}$ is characteristic.

$\sigma \in \mathrm{Aut}(G)$ is defined by $\sigma(1,\bar{0})$ and $\sigma(0,\bar{1})$ since:

$$\sigma(m, \bar{x}) = \sigma(m,0) + \sigma(0,\bar{x}) = m \cdot \sigma(1,0) + x \cdot \sigma(0,\bar{1}).$$

The choices for $\sigma(1,\bar{0})$ are $(1,\bar{0})$ or $(0,\bar{x})$, but the choice for $\sigma(0,\bar{1})$ is $(0,\bar{x})$ for any $\bar{x} \in \mathbb{Z}/p\mathbb{Z}$. Hence $\sigma(\{0\} \times \mathbb{Z}/p\mathbb{Z}) = \{0\} \times \mathbb{Z}/p\mathbb{Z}$ for all $\sigma \in \mathrm{Aut}(G)$.

Correspondence principle: A normal subgroup is exactly the kernel of some group homomorphism.

Clearly the kernels of group homomorphisms are normal. Now consider $N \trianglelefteq G$. Then:

$\bar{\ } : G \longrightarrow G/N$ is a group homomorphism with kernel exactly N.

$\varphi : G \longrightarrow H$, then $G/\ker(\varphi) \cong \mathrm{im}(\varphi)$ so we can rewrite $\bar{\varphi} : G \longrightarrow \mathrm{im}(\varphi) \cong G/\ker(\varphi)$.

The correspondence is between:

$$\{\text{subgroups of } G \text{ containing } N\} \xleftrightarrow{\text{bijection}} \{\text{normal subgroups of } G/N\}.$$

normal

$$H \xrightarrow{\alpha} H/N$$

$$H \xleftarrow{\beta} L = H/N$$

(1) Proving β exists, or equivalently that α is surjective.

(2) This bijection preserves normality.

(3) Proving that β is well defined (if $H_1/N = H_2/N$ then $H_1 = H_2$), or equivalently α is injective.

(4) Counting size.