HW 6.6.: Let G be a finite group and $p \in \mathbb{Z}^+$ the smallest prime dividing the order of G. If H is a subgroup of G of index p, show that $H \trianglelefteq G$.

Consider the group homomorphism
$$f : G \longrightarrow \Sigma(G/H) \qquad g \longmapsto \begin{pmatrix} f_g : G/H \to G/H \\ xH \mapsto gxH \end{pmatrix}$$
Since $|G/H| = [G:H] = p$, we have $\Sigma(G/H) \cong S_p$ the symmetric group on p elements. Now $\ker(f) \trianglelefteq G$, and $\ker(f) \subseteq H$ because if $g \in \ker(f)$ then $f(g) = \mathrm{id}_{\Sigma(G/H)}$ so $gH = f_g(H) = \mathrm{id}_{\Sigma(G/H)}(H) = H$ so $g \in H$.

Now $G/\ker(f) \cong f(G)$ a subgroup of S_p, so $[G:\ker(f)] = \left|G/\ker(f)\right|$ divides $|S_p| = p!$. Also $|G| = [G:\ker(f)]|\ker(f)|$, so $[G:\ker(f)]$ divides $|G|$. Since p is the smallest prime dividing $|G|$, we have that $[G:\ker(f)]$ is 1 or p. Since $\ker(f) \subseteq H \subsetneq G$ we have $[G:\ker(f)] \neq 1$ so $[G:\ker(f)] = p$. Now $\underbrace{[G:\ker(f)]}_{p} = \underbrace{[G:H]}_{p}[H:\ker(f)]$ so $[H:\ker(f)] = 1$ so $H = \ker(f) \trianglelefteq G$.