**HW7. Problem 5:** Let $G$ be a group, $|G|<60$, $|G|$ is not a prime. Then $G$ has a proper normal subgroup.

With the available tools that we have, we need to do this by cases. However, we can first reduce the number of cases we have to discuss. For this, we prove:

1. If $|G|=p^n$, $p$ prime, $n>1$, then $G$ has a proper normal subgroup.

   (this is HW7, Problem 1)

2. If $|G|=p\cdot q$, $p$ and $q$ prime, then $G$ has a proper normal subgroup.

3. If $|G|=p^2\cdot q$, $p$ and $q$ prime, then $G$ has a proper normal subgroup.

Now this reduces checking every order below 60 to just checking the orders 24, 30, 36, 40, 42, 48, 54, 56. We can use the Third Sylow Theorem and a straightforward element counting argument to prove that groups of orders 30, 40, 42, 54, 56 have a proper normal subgroup. To prove that groups of orders 24, 36, 48 have a proper normal subgroup is harder and requires explicit use of group actions.