Lemma: Let G be a finite group, P a Sylow p-subgroup, H a subgroup of G that is a p-group and $H \leq N_G(P)$ then $H \leq P$. In particular if H is a Sylow p-subgroup, then $H = P$.

Lemma: Let G be a finite group, P a Sylow p-subgroup, H a subgroup of G that is a p-group. Then:

(1) $C(P)$ consists of Sylow p-subgroups.

(2) The set $C(P)$ is an H-set by conjugation:
$$*: H \times C(P) \longrightarrow C(P).$$
$$(x, W) \longmapsto xWx^{-1}$$

(3) If T is a fixed point under the H-action, namely:

$$T \in F_H(C(P)) = \{W \in C(P) \mid xWx^{-1} = W \text{ for all } x \in H\},$$ then $H \subseteq T$.

(4) If H is a Sylow p-subgroup then H is the only possible fixed point, namely $F_H(C(P)) \subseteq \{H\}$.

Proof:

(3) Let T be a fixed point under the H-action, then the orbit $H * T = \{T\}$, so $xTx^{-1} = T$ for all $x \in H$. By definition, this means $H \subseteq N_G(T)$. Since $T \in C(P)$ then T is a Sylow p-subgroup, so by the previous Lemma $H \subseteq T$.

(4) If $T \in F_H(C(P))$ then $H \subseteq T$ by the above. Now T is a Sylow p-subgroup containing H which is also a Sylow p-subgroup, so $|H| = |T|$ so $H = T$.