MATH 141 Class Notes

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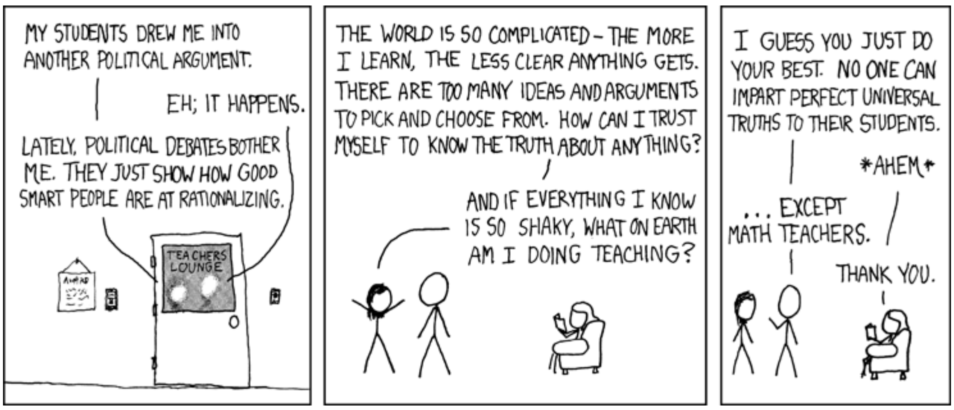
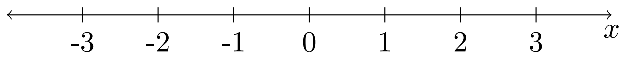
Contents


MY STUDENTS DREW ME INTO ANOTHER POLITICAL ARGUMENT.
EH; IT HAPPENS.
LATELY, POLITICAL DEBATES BOTHER ME. THEY JUST SHOW HOW GOOD SMART PEOPLE ARE AT RATIONALIZING.
TEACHERS LOUNGE
THE WORLD IS SO COMPLICATED – THE MORE I LEARN, THE LESS CLEAR ANYTHING GETS. THERE ARE TOO MANY IDEAS AND ARGUMENTS TO PICK AND CHOOSE FROM. HOW CAN I TRUST MYSELF TO KNOW THE TRUTH ABOUT ANYTHING?
AND IF EVERYTHING I KNOW IS SO SHAKY, WHAT ON EARTH AM I DOING TEACHING?
I GUESS YOU JUST DO YOUR BEST. NO ONE CAN IMPART PERFECT UNIVERSAL TRUTHS TO THEIR STUDENTS.
AHEM
... EXCEPT MATH TEACHERS.
THANK YOU.

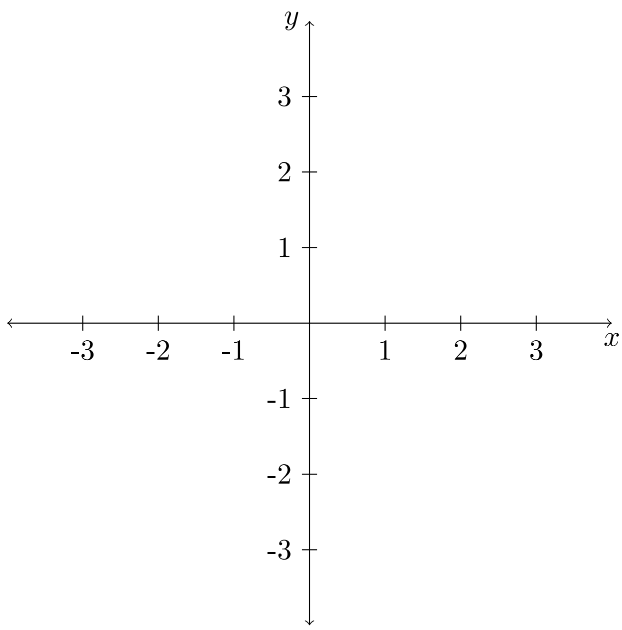
1 Straight Lines and Linear Functions

1.1 The Cartesian Coordinate System

The collection of all rational and irrational numbers is called the *real numbers*, and will be denoted by $\mathbb{R}$. The real numbers may be represented geometrically by points on a straight horizontal line, the so called *real line*: fix a point in the line to represent the number 0, also called *origin*, and then select another point to its right to represent the number 1, thus determining the scale for that number line. Positive numbers lie to the right of the origin, and negative numbers lie to the left of the origin.



The *Cartesian coordinate system* is a way of representing points in a plane: we take two perpendicular copies of the real line, making them intersect at their respective origins, and having the positive numbers be above and to the right of this intersection, while the negative numbers are below and to the left of the intersection. The horizontal line is called the *x axis* and the vertical line is called the *y axis*.



A point in the Cartesian coordinate system is represented by an ordered pair (x, y) or *coordinate*, where x or *abscissa* and y or *ordinate* correspond to distances in the respective axis. The axes divide the plane in four quadrants: $(+, +)$, $(-, +)$, $(-, -)$, and $(+, -)$, which are called *first*, *second*, *third*, and *fourth* quadrants respectively.

Definition 1.1. *The* distance d *between two points* $P_1(x_1, y_1)$ *and* $P_2(x_2, y_2)$ *in the plane is given by:*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 1.2. Draw in the plane and find the distance between the points:

1. $(4, 3)$ and $(2, 6)$,
2. $(1, 3)$ and $(4, 7)$,
3. $(1, 0)$ and $(4, 4)$.

Example 1.3. A marine biology experimental station is located on an island at the point $M = (0, 3000)$, while a power relay station is located on a straight coastal highway at the point $S = (10000, 0)$. A cable passing through the point $Q = (2000, 0)$ is to be laid to connect both stations. Assume that the x axis is the straight coastal highway, that the cost of running the cable on land is \$1.50 per unit, and that the cost on water is \$2.50 per unit. Draw a sketch of the situation and find the total cost for laying the cable.

Proposition 1.4. *The equation of a circle with center $C(h, k)$ and radius r is given by:*

$$(x - h)^2 + (y - k)^2 = r^2.$$

Example 1.5. A furniture store offers free setup and delivery services to all points within a $25km$ radius of its warehouse distribution center. If a house is located $20km$ east and $14km$ south of the warehouse, does it incur a delivery charge? Draw a sketch of the situation and justify your answer.

1.2 Straight Lines

A *straight line* L is uniquely determined by two distinct points (x_1, y_1) and (x_2, y_2). If $y_1 = y_2$ we say that the line is *horizontal*. If $x_1 = x_2$ we say that the line is *vertical*. If $x_1 \neq x_2$ we say that the line has a *slope*

Definition 1.6. *The* slope m *of* L *a non-vertical straight line* L *determined by two distinct points* (x_1, y_1) *and* (x_2, y_2) *is defined as:*

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Proposition 1.7. *Let L be a non-vertical straight line. Show that the slope m of L is independent of the two distinct points used to compute it, that is, if we have two* *pairs of distinct points* (x_1, y_1) *and* (x_2, y_2), *and* (x_1', y_1') *and* (x_2', y_2'), *then:*

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2' - y_1'}{x_2' - x_1'}.$$

The slope of a straight line is then a constant whenever it is defined. Interpreting $y_2 - y_1$ as the vertical change in y and $x_2 - x_1$ as the horizontal change in x, the slope m is a measure of the rate of change of y with respect to x.

Example 1.8. Sketch the straight line passing through the point $(-2, 5)$ with slope $-4/3$.

Example 1.9. Find the slope m of the line passing through the points $(-1, 1)$ and $(5, 3)$.

Definition 1.10. *We say that two distinct lines L_1 and L_2 are* parallel *whenever $m_1 = m_2$ (their slopes are equal) or they are both vertical.*

Example 1.11. Let L_1 be a line passing through the points $(-2, 9)$ and $(1, 3)$. Let L_2 be a line passing through the points $(-4, 10)$ and $(3, -4)$. Draw a sketch of these lines and determine whether they are parallel or not.

Definition 1.12. *A line L with slope m and passing through the point (x_1, y_1) is represented by the equation:*

$$y - y_1 = m(x - x_1).$$

This equation is called the point-slope form *of the equation of the line L.*

Example 1.13. Find an equation of the line that passes through the point $(1, 3)$ and has slope 2.

Example 1.14. Find an equation of the line passing through the points $(-3, 2)$ and $(4, -1)$.

Proposition 1.15. *Let L be a straight line lying in the xy plane. Then L can be represented by an equation involving the variables x and y.*

Definition 1.16. *Let L_1 and L_2 be two distinct non-vertical lines with slopes m_1 and m_2 respectively, we say that they are* perpendicular *whenever:*

$$m_1 = \frac{-1}{m_2}.$$

Example 1.17. Find an equation of the line that passes through the point $(3, 1)$ and is perpendicular to the line of Example 1.13.

A straight line L that is not horizontal nor vertical cuts the x axis and the y axis, say at points $(a, 0)$ and $(0, b)$ respectively. We call a the x *intercept* of L, and we call b the y *intercept* of L.

Definition 1.18. *The* slope-intercept form *of a line L is given by:*

$$y = mx + b$$

where L has slope m and y intercept b.

Proposition 1.19. *Let L be a line with slope m and y intercept b. Then the point-slope form of L coincides with the slope-intercept form of L.*

Example 1.20. Find an equation of the line that has slope 3 and y intercept -4.

Example 1.21. Determine the slope and y intercept of the line whose equation is $3x - 4y = 8$.

Example 1.22. A painting was purchased for \$50000 and it is expected to appreciate in value at a constant rate of \$5000 per year for the next 5 years. Write an equation predicting the value of the painting in the next several years. What will be its value 3 years from the purchase date?

Definition 1.23. *A line L is represented by the equation:*

$$Ax + By + C = 0$$

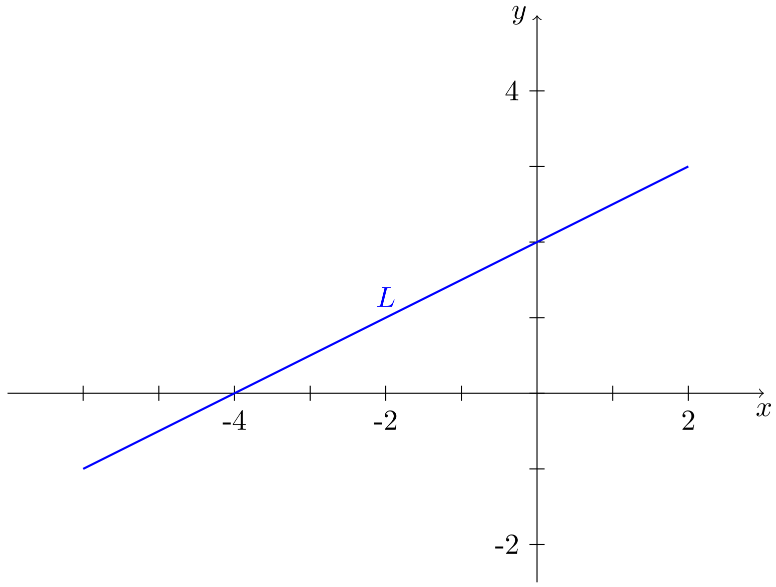
where A, B, and C are constants, and A, B are not both zero. This is called the general form *of a linear equation.*

Theorem 1.24. 1. *The equation of a straight line is a linear equation.*

2. *Every linear equation represents a straight line.*

Example 1.25. Draw a sketch of the straight line represented by the equation $3x - 4y - 12 = 0$.

Example 1.26. Find the slope, the point-slope form, the slope-intercept form, and the general form, of the following line:



1.3 Linear Functions and Mathematical Models

When applying Mathematics to solve real-world problems, we first need to formulate these problems in the language of Mathematics. This process is called *mathematical modeling*.

One of the most useful approximations of a real-world problem is via a *linear function*.

Definition 1.27. *A* function f *is a rule that assigns to each value of* x *one and only one value of* y*, and we denote this by* $y = f(x)$*. Given a function* f *the variable* x *is called the* independent variable *and the variable* y *is called the* dependent variable.

The collection X of all values that x may take is called the domain *of f. The collection Y of all values that y may take is called the* range *of f.*

We denote by $f : X \longrightarrow Y$ a function f with domain X and range Y.

Definition 1.28. *The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by:*

$$f(x) = mx + b$$

for some constants m and b is called a linear function.

Proposition 1.29. *Given f and g two linear functions, exactly one of the following happens: f is parallel to g but they not intersect, f and g are the same line, f and g intersect in a unique point.*

Proposition 1.30. *Given f and g two linear functions, then:*

$$c_1 f(x) + c_2 g(x) + c_3$$

for any constants c_1, c_2, and c_3 is also a linear function.

Given f and g two linear functions, their product $f(x)g(x)$ is not always a linear function: it suffices to take $f(x) = x = g(x)$ to obtain $f(x)g(x) = x^2$, which is already simplified and does not have the form in Definition 1.28.

Example 1.31. A truck has an original value of \$100000 and is to be depreciated linearly over 5 years with a \$30000 scrap value. Find an expression giving the book value at the end of year t and draw a sketch of it. What will be the book value of the truck at the end of the second year? What is the rate of depreciation of the truck? What will be the book value of the truck at the end of the sixth year?

When treating real-world problems, there are some special functions that provide the management of a business with a measure of certain relevant quantities.

Definition 1.32. *Let x denote the number of units of a product and p the price per unit of the product.*

1. *The* total cost function *is $C(x)$, the total cost of manufacturing x units of the product.*
2. *The* revenue function *is $R(x)$, the total revenue realized from the sale of x units of the product.*
3. *The* profit function *is $P(x)$ the total profit realized from manufacturing and selling x units of the product. In particular $P(x) = R(x) - C(x)$.*
4. *The* demand function *is an equation $p = D(x)$ expressing the relationship between the unit's price and the quantity demanded.*
5. *The* supply function *is an equation $p = S(x)$ expressing the relationship between the unit's price and the quantity supplied.*

Example 1.33. A manufacturer of iPhone cases has a monthly fixed cost of \$15000, a production cost of \$28 per case, and a selling price of \$51 per case. Find the cost function, the revenue function, and the profit function per case.

Example 1.34. When a pair of Nike shoes at Academy are priced at \$65, their demand is of 1000 units. When they are priced at \$85, their demand is of 600 units. Given that the demand is linear, find the demand equation. Above what price will there be no demand? What quantity would be demanded if the shoes were free?

Example 1.35. Hamilton Beach will make available 12000 blenders when the unit price is \$100. At a unit price of \$150, Hamilton Beach will market 16000 blenders. Knowing that it is known to be linear, find the equation relating the unit price of a blender to the quantity supplied. How many blenders will be marketed when the unit price is \$200? What is the lowest price at which a blender will be marketed?

1.4 Intersection of Straight Lines

Given two straight lines L_1 and L_2, they are represented by two linear functions that we will also denote by f_1 and f_2. As we have seen in Proposition 1.29, given f_1 and f_2 they can either be parallel, be the same line, or intersect in a unique point.

Definition 1.36. *Whenever f_1 and f_2 are the same, we say that there are an* infinite amount of points of intersection. *Whenever f_2 and f_2 are parallel, we say that there are* no points of intersection. *Whenever f_1 and f_2 intersect in a single point, we say that there is a* unique point of intersection.

Proposition 1.37. *Given two linear functions $f_1(x) = m_1x + b_1$ and $f_2(x) = m_2x + b_2$ with a unique point of intersection, that point can be written as:*

$$\left(\frac{b_2 - b_1}{m_1 - m_2}, \frac{m_1}{m_1 - m_1}(b_2 - b_1) + b_1\right) \quad \textit{and} \quad \left(\frac{b_2 - b_1}{m_1 - m_2}, \frac{m_2}{m_1 - m_1}(b_2 - b_1) + b_2\right).$$

To be safe, we can also check that:

$$\frac{m_1}{m_1 - m_1}(b_2 - b_1) + b_1 = \frac{m_1}{m_1 - m_1}(b_2 - b_1) + \frac{m_1 - m_2}{m_1 - m_2}b_1 = \frac{m_1 b_2}{m_1 - m_2} - \frac{m_2 b_1}{m_1 - m_2}$$
$$\frac{m_2}{m_1 - m_1}(b_2 - b_1) + b_2 = \frac{m_2}{m_1 - m_1}(b_2 - b_1) + \frac{m_1 - m_2}{m_1 - m_2}b_2 = \frac{-m_2 b_1}{m_1 - m_2} + \frac{m_1 b_2}{m_1 - m_2}$$

which are indeed the same.

Example 1.38. Draw a sketch and find the points of intersection of each of the following pairs of lines:

1. $2x - 3y = 6$ and $3x + 6y = 16$.
2. $y = 3x - 4$ and $x + 3y + 3 = 0$.

Example 1.39. Johnson & Johnson manufactures rubbing oil at a cost of \$4 per bottle, sells them for \$10 per bottle, and has a fixed production cost of \$9000 per month. Determine the break-even point of the manufacture of rubbing oil. What is the loss sustained by the firm if only 1200 bottles are produced and sold in a month? What is the profit if 2400 bottles are produced and sold in a month? How many units should the firm produce in order to realize a minimum profit of \$8000 in a month?

Example 1.40. Let L_1 and L_2 be two non vertical straight lines in the plane, with equations $f_1(x) = m_1x + b_1$ and $f_2(x) = m_2x + b_2$ respectively. Find conditions on m_1, m_2, b_1, and b_2 so that:

1. L_1 and L_2 do not intersect.
2. L_1 and L_2 intersect at exactly one point.
3. L_1 and L_2 intersect at infinitely many points.

Example 1.41. The Converse All-Stars can be assembled using machines ShoeMaker1 or ShoeMaker2. Converse estimates that the monthly fixed costs of using ShoeMaker1 are \$18000, whereas the monthly fixed costs of using ShoeMaker2 are \$15000. The variable costs of assembling a pair of All-Stars using ShoeMaker1 and ShoeMaker2 are \$15 and \$20 respectively. The shoes are sold for \$50 a pair. Find the cost functions associated to using each machine. Draw a sketch of the graphs of the cost functions and the revenue functions, use one axis for each machine. Which machine should management choose to maximize their profit if the projected sales are 450 pairs of shoes, and what would the profit be? And 550 pairs? And 650 pairs?

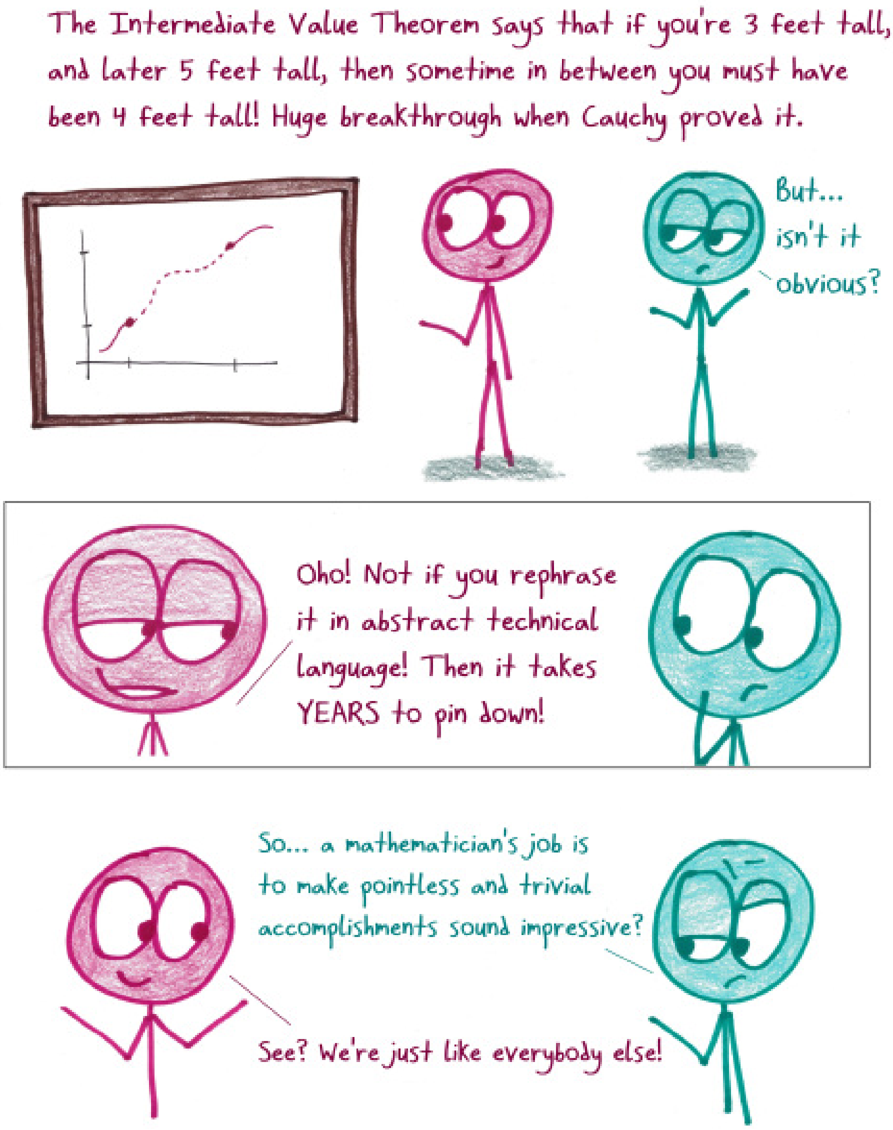
Example 1.42. Find conditions on a_1, a_2, b_1, b_2, c_1, and c_2 so that the system of linear equations:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

1. Has no solution.
2. Has a unique solution.
3. Has infinitely many solutions.

Example 1.43. The Nespresso coffee maker has a demand of 8000 units when each costs \$260. If each costs \$200, the demand increases to 10000 units. Nespresso will not market its coffee maker if the price is \$100 or lower, however, for each \$50 increase in the price above \$100, they will market an additional 1000 units. Assuming both the demand and supply equations are linear:

1. Find the demand equation.
2. Find the supply equation.
3. Find the equilibrium quantity and price.



The Intermediate Value Theorem says that if you're 3 feet tall, and later 5 feet tall, then sometime in between you must have been 4 feet tall! Huge breakthrough when Cauchy proved it.
But... isn't it obvious?
Oho! Not if you rephrase it in abstract technical language! Then it takes YEARS to pin down!
So... a mathematician's job is to make pointless and trivial accomplishments sound impressive?
See? We're just like everybody else!

2 Systems of Linear Equations and Matrices

2.1 Systems of Linear Equations: An Introduction

When we solve linear equations simultaneously, for example when finding break-even points or equilibrium points, we are solving *systems of linear equations*. We are not restricted to only two variables, we may have more. The following definition generalizes Definition 1.28 in that case.

Definition 2.1. *The function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by:*

$$f(x_1, \dots, x_n) = a_1x_1 + \cdots + a_nx_n + b$$

for some constants $a_1, \dots, a_n, b \in \mathbb{R}$ is called a linear function in n variables*.*

Definition 2.2. *A* system of m linear equations in n variables *is a collection of m linear functions in n variables:*

$$\begin{aligned} f_1(x_1, \dots, x_n) &= a_{11}x_1 + \cdots + a_{1n}x_n + b_1 = 0 \\ &\vdots \\ f_m(x_1, \dots, x_n) &= a_{m1}x_1 + \cdots + a_{mn}x_n + b_m = 0 \end{aligned}$$

for some constants $a_{11}, \dots, a_{1n}, b_1, \cdots, a_{m1}, \dots, a_{mn}, b_m \in \mathbb{R}$.

We number first rows, then columns: the constant a_{ij} is in f_i, accompanying x_j.

Proposition 2.3. *Given f and g two n linear functions, then:*

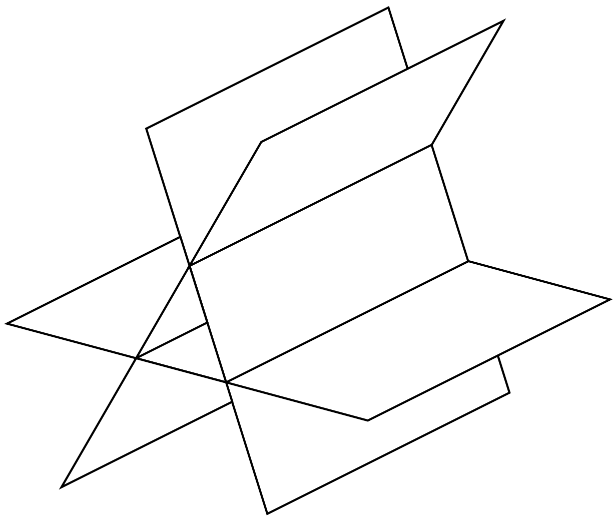
$$c_1f(x_1, \dots, x_n) + c_2g(x_1, \dots, x_n) + c_3$$

for any constants c_1, c_2, and c_3 is also an n linear function.

This is similar to Proposition 1.30 with the bookkeeping of the additional variables.

Definition 2.4. *A* solution of a system of m linear equations in n variables *is a collection of n constants $c_1, \dots, c_n$ satisfying all the individual equations simultaneously.*

However, we may still have situations where solutions do no exist. The following is a graphical representation of a system of three equations in three variables where we have infinitely many solutions for any pair of equations, but no solutions as a whole.



Proposition 2.5. *Given a system of m linear equations in n variables and $c_1, \ldots, c_n$ a solution to that system, if we either:*

1. *replace a function of the defining system by its multiplication by a non-zero constant,*

2. *replace a function of the defining system by its addition by any other linear function defining the system,*

then $c_1, \ldots, c_n$ remains a solution to that system.

Whenever we are faced with a system of two linear equations in two variables, there are two standard ways of solving it. Before starting to solve the system, we always want to start with a simplified system, meaning the equations need to be in general form. This is always a step that is necessary for both methods.

The *substitution method* is as follows:

1. Choose one equation and one variable. Solve that equation for that variable: put that variable to one side of the equal sign and everything else to the other side.

2. Substitute what was found in the other equation. This now becomes one equation in one variable. Solve this equation.

3. Substitute what was obtained in the original chosen equation, this gives a solution for the other variable.

The *elimination or addition method* is as follows:

1. Multiply the equations by numbers such that we obtain opposite coefficients of either the x or y variables. Add both equations.

2. We obtain an equation in one variable. Solve this equation.
3. Substitute what was obtained in either original equation of the system. This now becomes one equation in the remaining variable. Solve this equation.

However, we need to always keep in mind that in virtue of Proposition 1.29 our system of equations may have one solution, an infinite number of solutions, or no solutions. The methods above need to be interpreted in case our system of equations does not have a single solution.

Example 2.6. Solve the system of equations:

$$\begin{aligned} 2x - y &= 1 \\ 3x + 2y &= 12 \end{aligned}$$

Example 2.7. Solve the system of equations:

$$\begin{aligned} 2x - y &= 1 \\ 6x - 3y &= 3 \end{aligned}$$

Example 2.8. Solve the system of equations:

$$\begin{aligned} 2x - y &= 1 \\ 6x - 3y &= 12 \end{aligned}$$

Example 2.9. The total number of passengers riding the Brazos Transit District bus during the morning is 2746. If the child's fare is \$1.25, the adult fare is \$1.75, and the total revenue from the fares during the morning is \$4646, how many children and how many adults rode the bus during the morning?

2.2 Systems of Linear Equations: Unique Solutions

The methods described for two linear equations with two variables can be generalized for any number of equations and variables.

Example 2.10. Solve the system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

However, when the number of equations and variables increases, there are faster methods. For this, we need some nomenclature.

Definition 2.11. *A* matrix *is an ordered rectangular array of numbers. A matrix with m rows and n columns is said to have size $m \times n$. We denote by a_{ij} the entry in the i-th row and j-th column. We then write:*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

Notice how this closely resembles the notation in Definition 2.1. This is not a coincidence.

Definition 2.12. *Given a system of m linear equations in n variables:*

$$\begin{aligned} f_1(x_1, \dots, x_n) &= a_{11}x_1 \cdots + a_{1n}x_n + b_1 = 0 \\ &\vdots \\ f_m(x_1, \dots, x_n) &= a_{m1}x_1 \cdots + a_{mn}x_n + b_m = 0 \end{aligned}$$

we say that the coefficient matrix *of the linear system is:*

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix},$$

we say that the column of constants *of the linear system is:*

$$\begin{bmatrix} -b_1 \\ \vdots \\ -b_m \end{bmatrix},$$

we say that the augmented matrix *of the linear system is:*

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & -b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & -b_m \end{array}\right].$$

Example 2.13. Give the coefficient matrix, the column of constants, and the augmented matrix, of each of the following systems of equations:

$$\begin{aligned} 8x - 14y + 6z &= 21 \\ 3x + y + 5z &= 47 \\ 7x - 2y + z &= 13 \end{aligned} \quad , \quad \begin{aligned} x + 9y - 12z &= 3 \\ 7x + y - 6z &= 8 \\ 2x - 5y + 4z &= 2 \end{aligned}$$

Definition 2.14. *We say that a matrix is in* row-reduced form *if it satisfies all of the following conditions:*

1. *Each row consisting entirely of zeros lies below any other row having non-zero entries.*

2. *The first non-zero entry in each row is* 1.

3. *In any two successive non-zero rows, the leading* 1 *in the lower row lies to the right of the leading* 1 *in the upper row.*

4. *If a column containg a leading* 1*, then the other entries in that column are zeros.*

Example 2.15. Determine which of the following matrices are in row-reduced form. If a matrix is not in row-reduced form, state which condition is violated.

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 3\end{array}\right], \quad \left[\begin{array}{ccc|c}0 & 1 & 2 & -2\\ 1 & 0 & 0 & 3\\ 0 & 0 & 1 & 2\end{array}\right], \quad \left[\begin{array}{ccc|c}1 & 2 & 0 & 0\\ 0 & 0 & 1 & 3\\ 0 & 0 & 2 & 1\end{array}\right], \quad \left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\ 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right], \quad \left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 3\\ 0 & 1 & 0 & 2\end{array}\right]$$

Definition 2.16. *We call* elementary row operations *to the following row operations:*

1. *Interchange any two rows. We denote by $R_i \leftrightarrow R_j$ the interchange of row i and j.*

2. *Replace any row by a non-zero constant multiple of itself. We denote by cR_i the replacement of row i with c times row i.*

3. *Replace any row by the sum of that row and a constant multiple of any other row. We denote by $R_i + cR_j$ the replacement of row i with the sum of row i and c times row j.*

As we will see in Section 2.5, the elementary row operations can themselves be expressed as matrices. A key aspect is that if a matrix is given by a system of equations, then Proposition 2.5 assures us that doing elementary row operations to the matrix does not change the solutions to the original system of equations. The elementary row operations can then be used to simplify the matrices encoding the information given by a system of equations.

Example 2.17. Given the system of equations:

$$\begin{aligned} 2x - y - z &= 0 \\ 3x + 2y + z &= 7 \\ x + 2y + 2z &= 5 \end{aligned}$$

write its augmented matrix. What is the augmented matrix after $R_2 \leftrightarrow R_3$? If we then do the elementary row operation $\frac{1}{2}R_1$, what is the augmented matrix? If we then do the elementary row operation $R_1 + \frac{2}{3}R_2$, what is the augmented matrix?

One of the most important things that elementary row operations allow us to do is to transform any column into a column with a single one (in any given position) and everything else zeros:

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{mj} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \xrightarrow{\frac{1}{a_{ij}}R_i, R_1 - a_{1j}R_i, \ldots, R_m - a_{mj}R_i} \begin{bmatrix} \cdots & 0 & \cdots \\ & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \\ \cdots & 0 & \cdots \end{bmatrix}.$$

This is called *pivoting* the matrix about the element a_{ij}.

Example 2.18. Pivot the given matrices about the desired elements:

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 5 & 6 & 2 & -4 \end{array}\right] \text{ about } a_{2,3}, \quad \left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 2 & 4 & 8 & 6 \\ -1 & 2 & 3 & 4 \end{array}\right] \text{ about } a_{2,1}, \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & -3 & 3 & 2 \\ 0 & 4 & -1 & 3 \end{array}\right] \text{ about } a_{2,2}.$$

Definition 2.19. *Given a linear system of equations, we call the* Gauss-Jordan elimination method *to the following algorithm:*

1. *Write the augmented matrix corresponding to the linear system.*
2. *Interchange rows, if necessary, to obtain an augmented matrix in which the first entry in the first row is non-zero. Pivot the matrix about this entry.*
3. *Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is non-zero. Pivot the matrix about this entry.*
4. *Continue until the matrix is in row-reduced form.*

If the system of linear equations has a unique solution, the Gauss-Jordan elimination method always yields the matrix with 1 in the diagonal, and the solution can be read directly from the matrix: each column corresponds to a variable, and looking at the only row that has a 1 for that column, its corresponding value is the rightmost element in said row. This is again consequence of Proposition2.5, since we are only doing elementary row operations.

Example 2.20. Solve the system of equations:

$$\begin{aligned} 2x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned}$$

using the Gauss-Jordan elimination method.

Example 2.21. Solve the system of equations:

$$\begin{aligned} 2x + y + z &= 180 \\ x + 3y + 2z &= 300 \\ 2x + y + 2z &= 240 \end{aligned}$$

using the Gauss-Jordan elimination method.

Example 2.22. Solve the system of equations:

$$\begin{aligned} 2x + 3y + z &= 6 \\ x - 2y + 3z &= -3 \\ 3x + 2y - 4z &= 12 \end{aligned}$$

using the Gauss-Jordan elimination method.

Example 2.23. The Texas A&M Library System edits three types of notebooks: A3, A4, and A5. Each of the three campuses, namely College Station, Galveston, and Qatar, requires a different amount of minutes to review a dozen notebooks, and a notebook is only approved after all campuses have reviewed it. The information is as shown on the following table:

	A3	A4	A5
College Station	9	12	15
Galveston	22	24	28
Qatar	6	8	8

if each campus has available a maximum of 80, 160, and 48 labor minutes, respectively, per day, how many dozens of each type of notebook can be reviewed each day if the editors work at full capacity?

2.3 Systems of Linear Equations: Undetermined and Overdetermined Systems

The Gauss-Jordan elimination method can also be applied if the system of linear equations has infinitely many solutions, or no solutions. However, in those cases we have to be careful and interpret the information encoded in the augmented matrix.

Some of the pathologies that we may encounter include having rows full of zeros, a zero row in the matrix of coefficients with a non-zero in the column of constants, more equations than variables, and more variables than equations.

Example 2.24. Solve the system of equations:

$$\begin{aligned} x + 2y - 3z &= -2 \\ 3x - y - 2z &= 1 \\ 2x + 3y - 5z &= -3 \end{aligned}$$

using the Gauss-Jordan elimination method.

Example 2.25. Solve the system of equations:

$$\begin{aligned} x + y + z &= -2 \\ 3x - y - z &= 4 \\ x + 5y + 5z &= -1 \end{aligned}$$

using the Gauss-Jordan elimination method.

Example 2.26. Solve the system of equations:

$$\begin{aligned} x + 2y &= 4 \\ x - 2y &= 0 \\ 4x + 3y &= 12 \end{aligned}$$

using the Gauss-Jordan elimination method.

Example 2.27. Solve the system of equations:

$$\begin{aligned} x + 2y - 3z + w &= -2 \\ 3x - y - 2z - 4w &= 1 \\ 2x + 3y - 5z + w &= -3 \end{aligned}$$

using the Gauss-Jordan elimination method.

Example 2.28. The management of Enterprise has allocated \$840000 to purchase 60 new cars to add to their existing fleet of rental cars. The company will choose from compact, mid sized, and large cars, costing \$10000, \$16000, and \$22000 each, respectively. Find formulas giving the options available to Enterprise. Give two different specific purchasing options.

2.4 Matrices

We can solve problems by realizing operations on matrices representing the data of said problems. For this we should be able to enter data into matrix form, and also to interpret the entries in a matrix as giving us specific information.

Example 2.29. In May 2019, the company Marshall produced four different loudspeaker systems, Acton 1, Acton 2, Stanmore 1, and Stanmore 2, in three separate locations, Dundee, Edinburgh, and London, as shown in the following table:

	Acton 1	Acton 2	Stanmore 1	Stanmore 2
Dundee	320	280	460	280
Edinburgh	480	360	580	0
London	540	420	200	880

We can summarize the data in the following production matrix P:

$$P = \begin{bmatrix} 320 & 280 & 460 & 280 \\ 480 & 360 & 580 & 0 \\ 540 & 420 & 200 & 880 \end{bmatrix}$$

where the numbers in each row give the numbers of loudspeakers produced in the corresponding location, while the numbers in each column give the production of a particular model of loudspeaker in terms of the location where it was produced.

1. What is the size of the matrix P?
2. What is the value of a_{24}? What is its interpretation, that is, what does this number mean in terms of production of loudspeakers?
3. Find the sum of the entries of the first row of P and interpret the result in terms of production of loudspeakers.
4. Find the sum of the entries of the fourth column of P and interpret the result in terms of production of loudspeakers.

Definition 2.30. *We say that two matrices A and B are* equal *if they have the same size, say $m \times n$ and their corresponding entries are equal, that is, $a_{ij} = b_{ij}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.*

Definition 2.31. *Let A be a matrix of size $m \times n$ with entries a_{ij}, and c a real number. Then its* scalar product *cA is the matrix obtained by multiplying each entry of A by c, that is, with entries ca_{ij} for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.*

Definition 2.32. *Let A and B be two matrices of the same size, say $m \times n$. Their* sum *$A + B$ is the matrix obtained by adding the corresponding entries in the two matrices: $[a + b]_{ij} = a_{ij} + b_{ij}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Their* difference *$A - B$ is the matrix obtained by subtracting the corresponding entries in the two matrices: $[a - b]_{ij} = a_{ij} - b_{ij}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.*

Example 2.33. Consider the following matrices:

$$A = \begin{bmatrix} 4 & -5 & 6 \\ -11 & 2 & 4 \\ 3 & 6 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 5 & 1 \\ 4 & 15 & 8 \\ 2 & 9 & -1 \end{bmatrix},$$
$$C = \begin{bmatrix} -8 & 7 & 12 \\ 5 & -21 & 3 \\ -9 & 18 & -6 \end{bmatrix}, \quad D = \begin{bmatrix} 9 & -7 & -5 \\ -12 & 38 & 9 \\ 14 & -3 & 13 \end{bmatrix}.$$

1. Compute $A + B$, $A + C$, and $A + D$.
2. Compute $B + A$, $B + C$, and $B + D$.
3. Compute $C + A$, $C + B$, and $C + D$.
4. Compute $D + A$, $D + B$, and $D + C$.
5. Do we have $A + B = C + D$? Do we have $A + C = B + D$? Do we have $A + D = B + C$?

Proposition 2.34. *Let A and B be two matrices of the same size, say $m \times n$. Then $A + B = B + A$, that is, matrix addition is commutative.*

Proposition 2.35. *Let A, B, and C be three matrices of the same size, say $m \times n$. Then $A + (B + C) = (A + B) + C$, that is, matrix addition is associative.*

Definition 2.36. *Let A be a matrix of size $m \times n$ with entries a_{ij}, then its* transpose *A^T is the $n \times m$ matrix with elements a_{ij} for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.*

Example 2.37. Given:

$$A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$$

find the matrix X satisfying the matrix equation $2X + B = 3A$. What is the size of X?

Example 2.38. The Aggieland Credit Union offers three types of bank accounts: checking, saving, and fixed deposit. They have three branches: one in William Joel Bryan Parkway, one in University Drive, and one in Southwest Parkway. In August 2019, the freshmen opened the following accounts:

	Checking	Saving	Fixed deposit
W. J. Bryan Pkwy.	2820	1470	1120
University Dr.	1030	520	480
Southwest Pkwy.	1170	540	460

while (using the same column and rows as above) matrix B and C below represent the number of accounts opened and closed by freshmen during September 2019, respectively:

$$B = \begin{bmatrix} 260 & 120 & 110 \\ 140 & 60 & 50 \\ 120 & 70 & 50 \end{bmatrix}, \quad C = \begin{bmatrix} 120 & 80 & 80 \\ 70 & 30 & 40 \\ 60 & 20 & 40 \end{bmatrix}$$

1. Find matrix D representing the total number of each type of account at the name of a freshman at the end of September 2019 in each location.

2. It is expected that the number of accounts at each location increase by 10% during October 2019. Write matrix E representing this expected increase.

2.5 Multiplication of Matrices

We have already seen a few operations that can be performed on matrices, some of which required that the matrices had the same size. We can also define multiplication of matrices, but for this we require that the number of column of the first agrees with the number of rows of the second.

Definition 2.39. *Let A be a matrix of size $m \times n$ and B a matrix of size $n \times r$ for any natural numbers m, n, r. Their* product *AB is the matrix of size $m \times r$ obtained component-wise by:*

$$[AB]_{ij} = a_{i1}b_{1j} + a_{21}b_{2j} + \cdots + a_{i(n-1)}b_{(n-1)j} + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

for all $i = 1, \ldots, m$ and $j = 1, \ldots, r$.

That is, to obtain the entry (i, j) we multiply element-wise the i-th row of A by the j-th column of B, and then we add these multiplications.

Example 2.40. Let:

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 4 & -1 \\ 2 & 4 \end{bmatrix}$$

compute AB and BA.

If we use as an example A of size 2×3 and B of size 3×2, we can put the matrices in such a way that this multiplication becomes intuitive. The multiplication AB can be viewed as:

$$\begin{array}{cc} & \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} & \begin{bmatrix} \star & \star \\ \star & \star \end{bmatrix} \end{array}$$

while the multiplication BA can be viewed as:

$$\begin{array}{cc} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \\ \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} & \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix} \end{array}$$

Example 2.41. Let:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 2 & 3 \\ 3 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

compute AB and BA.

Proposition 2.42. *If the products and sums of three matrices A, B, and C are defined, then we have:*

1. *multiplication is associative:* $A(BC) = (AB)C$*,*
2. *multiplication distributes over addition:* $A(B + C) = AB + AC$*.*

Example 2.43. Write the following system of linear equations as an equality of matrices, and solve it:

$$\begin{aligned} 2x - 4y + z &= 6 \\ -3x + 6y - 5z &= -1 \\ x - 3y + 7z &= 0 \end{aligned}$$

Example 2.44. The skateboarding company enjoi is charged with an order of board decks consisting of 900 Deedz Avocado, 1200 Pilz Pills, and 2000 Raining Cats & Dogs. The company decides to manufacture 500 Deedz Avocado, 800 Pilz Pills, and 1300 Raining Cats & Dogs board decks in Los Angeles, while the rest of the order will be manufactured in San Francisco. Each board deck requires different specifications of grip tape, paint, and wood. Deedz Avocado requires 1.5 meters of grip tape, 30 square meters of paint, and 5 pieces of wood. Pilz Pills requires 2 meters of grip tape, 35 square meters of paint, and 8 pieces of wood. Raining Cats & Dogs requires 2.5 meters of grip tape, 25 square meters of paint, and 15 pieces of wood. The grip tape costs \$4.50 per meter, the paint costs \$0.10 cents per square meter, and the wood costs \$0.25 cents per piece.

1. Express in a 2×3 matrix P the production of types of board decks with respect to the location where each is produced.
2. Express in a 3×3 matrix A (for activity) the amount of materials required to manufacture each type of board deck.
3. Express in a 3×1 matrix C the cost of each type of material.
4. Find how much each type of material must be purchased for each location.
5. What is the total cost of materials incurred by each plant and the total cost of materials incurred by the enjoi skateboarding company in filling the order?

Example 2.45. For the diet of the Texas A&M football players, Jimbo Fisher gives them supplements for carbohydrates, proteins, and vitamins. The way they consume these supplements are in form of Jack Link's, Muscle Milk, and Powerade. The number of grams of supplement in each one of these products is given by the following table:

	Jack Link's	Muscle Milk	Powerade
Carbohydrates	400	1200	800
Proteins	110	570	340
Vitamins	90	30	60

Kellen Mond consumes 7 Jack Link's, 1 bottle of Muscle Milk, and 6 bottles of Powerade for lunch, and 9 Jack Link's, 3 bottles of Muscle Milk, and 2 bottles of Powerade for supper. Calculate the following in matrix form, and explain the meaning of the entries in each matrix:

1. The table multiplied by the transpose of Kellen Mond's lunch.
2. The table multiplied by the transpose of Kellen Mond's supper.
3. The table multiplied by the transpose of Kellen Mond's lunch and supper.

3 Linear Programming: A Geometric Approach

3.1 Graphing Systems of Linear Inequalities in Two Variables

In many world problems we are not concerned with equations, where we are trying to achieve equality, but instead we have a goal that we would like to attain and have some freedom on how to attain it. The mathematical representation of this is an inequality, and when multiple such conditions are imposed we are solving *systems of linear inequalities*. As in with systems of linear equations, we are not restricted to only two variables and we may have more. The following definition generalizes Definition 2.2.

Definition 3.1. *A* system of m linear inequalities in n variables *is a collection of m linear functions in n variables:*

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= a_{11}x_1 + \cdots + a_{1n}x_n + b_1 \geq 0 \\ &\vdots \\ f_m(x_1, \ldots, x_n) &= a_{m1}x_1 + \cdots + a_{mn}x_n + b_m \geq 0 \end{aligned}$$

for some constants $a_{11}, \ldots, a_{1n}, b_1, \cdots, a_{m1}, \ldots, a_{mn}, b_m \in \mathbb{R}$.

Where we again number first rows, then columns: the constant a_{ij} is in f_i, accompanying x_j. Notice that in particular the case where all inequalities are equal signs recovers the usual system of equations. Moreover, notice that this definition also includes the case $\leq$ since the inequality $f(x_1, \ldots, x_n) \leq 0$ is equivalent to the inequality $-f(x_1, \ldots, x_n) \geq 0$.

Definition 3.2. *A* solution of a system of m linear inequalities in n variables *is a collection of n constants $c_1, \ldots, c_n$ satisfying all the individual inequalities simultaneously.*

Example 3.3. Determine the solution set for the inequality $2x + 3y < 6$.

Proposition 3.4. *The following is a procedure for graphing linear inequalities:*

1. *Draw the graph of the equation obtained by replacing the inequality by an equality. Draw a dotted line if a strict inequality was replaced. Draw a solid line if not.*
2. *Pick a test point lying in one of the half planes that remain, and substitute the values of the point in the inequality.*
3. *If the inequality is satisfied, the solution to the inequality includes the half plane containing the test point. Otherwise, the solution to the inequality includes the half plane not containing the test point.*

Example 3.5. Determine the solution set for the inequality $x - 2y > 0$.

Example 3.6. Determine the solution set for the inequality $x \leq -1$.

This can also be applied to systems of linear inequalities. In this case, the solution is found by shading the corresponding regions for each inequality, and the region that is shaded under all inequalities will be the solution.

Definition 3.7. *A solution to a system of linear inequalities is called* bounded *if* *it can be enclosed by a circle of sufficiently large radius. Otherwise, it is called* unbounded.

Example 3.8. Determine the solution set for the system of inequalities:

$$\begin{aligned} 4x + 3y &\geq 12 \\ x - y &\leq 0 \end{aligned}$$

Example 3.9. Determine the solution set for the system of inequalities:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 6 \\ 2x + y &\leq 8 \end{aligned}$$

Example 3.10. Determine the solution set for the system of inequalities:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + y &\geq 50 \\ x + 2y &\leq 40 \end{aligned}$$

3.2 Linear Programming Problems

The solutions of systems of linear inequalities are usually plentiful. In virtue of Proposition 1.29 we may only have one solution, an infinite number of solutions, or no solutions when we consider the equality. But when considering inequalities, this range is likely to be expanded.

Suppose we have no solutions for the equality, then we have no solutions for the inequality. Suppose we have infinitely many solutions for the equality, then we have infinitely many solutions for the inequality. Suppose we have exactly one solution for the equality, then for the inequality each equation restricts us to a half plane (instead of restricting us to a single line like the equality does), so we are much more likely to have additional solutions.

Because of this, we look for certain special solutions to a system of inequalities.

Definition 3.11. *Consider a system of m linear inequalities in n variables:*

$$\begin{aligned} f_1(x_1,\ldots,x_n) &= a_{11}x_1+\cdots+a_{1n}x_n+b_1 \geq 0 \\ &\vdots \\ f_m(x_1,\ldots,x_n) &= a_{m1}x_1+\cdots+a_{mn}x_n+b_m \geq 0 \end{aligned}$$

where we want to optimize certain inequalities, say $f_i(x_1,\ldots,x_n)\geq 0$ for some i. We call objective functions *the functions that we want to optimize. We call* linear constraints *the functions that we do not want to optimize.*

Definition 3.12. *A* linear programming problem *consists of a linear objective function to be maximized or minimized subject to certain constraints in the form of linear equalities or inequalities.*

Example 3.13. Formulate but do not solve. The Texas A&M University fraternity PΩP (Rho Omega Rho) is selling pulled pork and smoked brisket for a fundraiser. Each slider of pulled pork will result in a profit of \$1 and each plate of smoked brisket will result in a profit of \$1.20. To prepare a slider they require 2 minute of cutting time and 1 minute of plating. To prepare a plate they require 1 minutes of cutting time and 3 minute of plating. The volunteers for cutting are available for 3 hours, and the volunteers for plating are available for 5 hours. How many sliders and how many plates should PΩP make in order to maximize their profit?

Example 3.14. Formulate but do not solve. A Texas A&M University sophomore is advised that they should study for Math 141 at least 2.4 hours for Chapter 1, 2.1 hours for Chapter 2, and 1.5 hours for Chapter 3. The ideal places of studying for this student are Sweet Eugene's and 1514 Pastries and Coffee. Whenever the student goes to Sweet Eugene's they are able to invest 0.4 hours in Chapter 1, 0.1 hours in Chapter 2, and 0.05 hours in Chapter 3, and their coffee amounts to \$2. Whenever the student goes to 1514 Pastries and Coffee they are able to invest 0.1 hours in Chapter 1, 0.15 hours in Chapter 2, and 0.15 hours in Chapter 3, and their coffee amounts to \$3. How many times should the student go to Sweet Eugene's and 1514 Pastries and Coffee in order to meet the required hours of study, if the student wants to spend the least possible amount of money on coffee?

Example 3.15. Formulate but do not solve. The United States Postal Service in the Brazos County has two main offices, one in Texas Avenue and the other in University Drive. The maximum volume of correspondence that these offices can manage is of 400 and 600 units per day, respectively. The correspondence is then shipped to three warehouses to be distributed, one in Bryan, one in College Station, and one in Steep Hollow. For the warehouses to meet their orders, the minimum daily requirements are of 200, 300, and 400 units, respectively. The shipping costs from Texas Ave. to Bryan, College Station, and Steep Hollow are of \$0.20, \$0.08, and \$0.10 per unit, respectively. The shipping costs from University Dr. to Bryan, College Station, and Steep Hollow are of \$0.12, \$0.22, and \$0.18 per unit, respectively. How should USPS organize the shipping to warehouses if they want to meet the requirements of the distribution centers and at the same time keep the shipping costs to a minimum?

3.3 Graphical Solution to Linear Programming Problems

When dealing with linear programming problems, there is a usual nomenclature used.

Definition 3.16. *Given a linear programming problem, consider the system of linear inequalities that form it. These inequalities define a region S on the plane. The set S is called a* feasible set*. Each point in S is called a* feasible solution*. The feasible solution that maximizes the objective function from the linear programming problem is called an* optimal solution*.*

Note that an optimal solution may not be unique. In particular, sometimes we encounter lines in our feasible set where all the points attain the same value on the objective function.

Definition 3.17. *Consider a linear programming problem. A line restricted to our feasible set having all its points corresponding to the same value on the objective function is called an* isoprofit line*.*

Theorem 3.18. *Consider a linear programming problem. If it has a solution, it must occur at a corner. Moreover if the objective function takes its maximum or minimum at two corners which lie on the same boundary line, then the objective function is constant on every point between those two corners.*

To solve linear programming problems, we use the *method of corners*, which is based on Theorem 3.18. Given a linear programming problem, we proceed as follows:

1. Graph the feasible set.
2. Find the coordinates of all corner points (these are the points of intersection of the constraints).
3. Evaluate the objective function at each corner.

Finally, to determine the optimal solution(s), find how many corners are there satisfying the conditions that we want over the objective function. If there is only one corner, then it constitutes a unique solution. If there are two corners, then all the points in the segment between them is a solution.

Example 3.19. Find the maximum and minimum of $P = 2x + 3y$ subject to the following system of inequalities:

$$\begin{aligned}
x &\geq 0 \\
y &\geq 0 \\
x &\leq 10 \\
x + y &\geq 5 \\
x - y &\leq 5 \\
2x + 3y &\leq 30
\end{aligned}$$

Example 3.20. The Texas A&M University fraternity PΩP (Rho Omega Rho) is selling pulled pork and smoked brisket for a fundraiser. Each slider of pulled pork will result in a profit of \$1 and each plate of smoked brisket will result in a profit of \$1.20. To prepare a slider they require 2 minute of cutting time and 1 minute of plating. To prepare a plate they require 1 minutes of cutting time and 3 minute of plating. The volunteers for cutting are available for 3 hours, and the volunteers for plating are available for 5 hours. How many sliders and how many plates should PΩP make in order to maximize their profit?

Example 3.21. A Texas A&M University sophomore is advised that they should study for Math 141 at least 2.4 hours for Chapter 1, 2.1 hours for Chapter 2, and 1.5 hours for Chapter 3. The ideal places of studying for this student are Sweet Eugene's and 1514 Pastries and Coffee. Whenever the student goes to Sweet Eugene's they are able to invest 0.4 hours in Chapter 1, 0.1 hours in Chapter 2, and 0.05 hours in Chapter 3, and their coffee amounts to \$2. Whenever the student goes to 1514 Pastries and Coffee they are able to invest 0.1 hours in Chapter 1, 0.15 hours in Chapter 2, and 0.15 hours in Chapter 3, and their coffee amounts to \$3. How many times should the student go to Sweet Eugene's and 1514 Pastries and Coffee in order to meet the required hours of study, if the student wants to spend the least possible amount of money on coffee?

5 Mathematics of Finance

In this chapter we will see how we can compute interests, payments on a mortgage, or which type of loan is more beneficial depending on our resources.

5.1 Compound Interest

Definition 5.1. *Given a certain amount P, the so called* original principal*, an interest rate of r per year, and an amount t of years, the interest that is computed over the original principal only is called* simple interest*.*

Proposition 5.2. *The accumulated amount A obtained by the simple interest of an original principal P at an interest rate of r per year for t years is:*

$$A = P(1 + rt).$$

Example 5.3. An amount of \$2000 is invested in a 10 year trust fund that pays 6% annual simple interest. What is the total amount of the trust fund at the end of 10 years?
Solution: The total amount of the trust fund at the end of 10 years is:

$$2000 + I = 2000 + 2000 \cdot 0.06 \cdot 10 = 3200.$$

Definition 5.4. *Given a certain amount of money P, the so called original principal, an interest rate of r per year, the so called* nominal rate*,compounded m times per year, and an amount t of years, the interest that is periodically added to the principal and itself also earns interest at the same rate is called* compound interest*.*

Proposition 5.5. *The accumulated amount A obtained by the compound interest of an original principal P at an interest rate of r per year, compounded m times a year, for t years is:*

$$A = P\left(1 + \frac{r}{m}\right)^{mt}.$$

Proposition 5.6. *The accumulated amount A obtained by the compound interest of an original principal P at an interest rate of r per year, compounded continuously, for t years is:*

$$A = Pe^{rt}.$$

Example 5.7. An account that pays 7.2% has \$5000. The money remains there for 9 years. How much interest is earned if it is compounded daily? And if it is compounded continuously?
Solution: If the interest is compounded daily the total amount in the account at the end of 9 years is:

$$5000\left(1+\frac{0.072}{365}\right)^{365\cdot 9}\approx 9557.96.$$

If the interest is compounded continuously the total amount in the account at the end of 9 years is:

$$5000e^{0.072\cdot 9}\approx 9558.57.$$

Example 5.8. Find the accumulated amount after 3 years of \$1000 invested at 8% per year compounded:

1. Annually,
2. Semiannually,
3. Quarterly,
4. Monthly,
5. Daily.

Solution: We have:

1. $A=1000\left(1+\frac{0.08}{1}\right)^{1\cdot 3}\approx 1259.71$,
2. $A=1000\left(1+\frac{0.08}{2}\right)^{2\cdot 3}\approx 1265.32$,
3. $A=1000\left(1+\frac{0.08}{4}\right)^{4\cdot 3}\approx 1268.24$,
4. $A=1000\left(1+\frac{0.08}{12}\right)^{12\cdot 3}\approx 1270.24$,
5. $A=1000\left(1+\frac{0.08}{365}\right)^{365\cdot 3}\approx 1271.22$.

Example 5.9. How much money should be deposited in a bank paying interest at the rate of 6% per year compounded monthly so that at the end of 3 years the accumulated amount will be \$20000?
Solution: We have:

$$20000=P\left(1+\frac{0.06}{12}\right)^{12\cdot 3}$$

so $P=16712.9$ dollars.

Example 5.10. We are given two investment options:

1. Purchase a vinyl that matures in 12 years and pays interest upon maturity at a rate of 10% per year compounded daily.
2. Purchase a zero coupon vinyl that will triple her investment in the same period.

Which will optimize our investment?
Solution: Setting P any initial payment that we had to make to purchase the vinyl, the first option gives us:

$$A_1 = P\left(1+\frac{0.1}{365}\right)^{365\cdot 12} \approx P \cdot 3.32$$

while the second option gives us:

$$A_2 = P \cdot 3$$

and thus since $A_1 \geq A_2$ the first option is a better investment.

Definition 5.11. *The* effective rate r_{eff} *is the simple interest rate that would produce the same accumulated amount in a year that the nominal rate r would produce when compounded m times in a year.*

Proposition 5.12. *The effective rate r_{eff} given by a nominal rate r compounded m times in a year is:*

$$r_{eff} = \left(1+\frac{r}{m}\right)^m - 1.$$

Example 5.13. Find the effective rate corresponding to a nominal rate of 8% per year compounded:

1. Annually,
2. Semiannually,
3. Quarterly,
4. Monthly,
5. Daily.

Solution: We have:

1. $r_{eff} = \left(1+\frac{0.08}{1}\right)^1 - 1 = 0.08$,
2. $r_{eff} = \left(1+\frac{0.08}{2}\right)^2 - 1 = 0.0816$,
3. $r_{eff} = \left(1+\frac{0.08}{4}\right)^4 - 1 \approx 0.0824$,

4. $r_{eff} = \left(1 + \frac{0.08}{12}\right)^{12} - 1 \approx 0.083$,

5. $r_{eff} = \left(1 + \frac{0.08}{365}\right)^{365} - 1 \approx 0.0833$.

Example 5.14. Wells Fargo advertises a nominal rate of 7.1% compounded semiannually. Chase advertises a nominal rate of 7% compounded daily. What are the effective yields? In which bank should one deposit their money?
Solution: The effective yield offered by Wells Fargo is $(1+0.071/2)^2 - 1 \approx 0.0723$. The effective yield offered by Chase is $(1+0.07/365)^{365} - 1 \approx 0.0725$. We should then deposit our money in Chase.

5.2 Annuities

Definition 5.15. *An* annuity *is a sequence of payments made at regular time intervals. The time period in which the payments are made is called the* term *of the annuity. The* future value of the annuity *is the total amount that was paid and the interest that those payments incurred in, at the end of the term of the annuity. The* present value of the annuity *is the total (that is, without regular payments) amount that, when invested at the same interest and for the same period of time as the annuity, yields the same accumulated value.*

We will consider annuities in which the time intervals are fixed, the payments are made at the end of each payment period, and the payment period coincides with the interest conversion period. That is, our annuities are subject to the following conditions:

1. The terms are given by fixed time intervals.

2. The periodic payments are equal in size.

3. The payments are made at the end of the payment period.

4. The payment periods coincide with the interest conversion periods.

Proposition 5.16. *The future value of an annuity, called S, of n payments of R dollars each, paid at the end of each investment period into an account that earns interest at the rate of i per period, is:*

$$S = R\left(\frac{(1+i)^n - 1}{i}\right).$$

Using that $i = r/m$ and $n = mt$ is the conversion between interest rate per period and nominal annual rate, when taking into account how many times a year and for how many years the interest is compounded, we also have:

Proposition 5.17. *The future value of an annuity, called S, of payments of R dollars each, paid at the end of each investment period into an account that earns a nominal annual rate of r compounded m times a year, for t years, is:*

$$S = R\left(\frac{\left(1+\frac{r}{m}\right)^{mt}-1}{r/m}\right).$$

Example 5.18. Find the amount of an ordinary annuity of 12 monthly payments of \$100 that earn interest at 12% per year compounded monthly.
Solution: We have $r = 0.12$, $m = 12$, $t = 1$, $R = 100$, and so:

$$S = \frac{100 \cdot ((1.01)^{12} - 1)}{0.01} \approx 1268.25.$$

Example 5.19. Sally deposits \$100 per month into an annuity at a nominal rate of 5% per year compounded monthly, for 9 years. Find the future value of the annuity and the amount of interest earned during this investment period.
Solution: We have $r = 0.05$, $m = 12$, $t = 9$, $R = 100$, and so:

$$S = \frac{100 \cdot ((1 + 0.05/12)^{9 \cdot 12} - 1)}{0.05/12} \approx 13604.32.$$

This means that the interest earned in dollars, which is the final value in the account minus the total amount paid into the account, is:

$$S - R \cdot m \cdot t =\approx 13604.32 - 108 \cdot 100 = 2804.32.$$

Example 5.20. Jennifer wants to save \$20000 for a down payment on a car after she graduates from college. She opens an annuity at a nominal rate of 2.25% per year compounded quarterly, for 5 years. How much does she need to pay quarterly to obtain the desired result?
Solution: We have $r = 0.025$, $m = 4$, $t = 5$, $S = 20000$, and so:

$$20000 = \frac{R \cdot ((1 + 0.025/4)^{5 \cdot 4} - 1)}{0.025/4} = 21.2332R$$

and thus $R \approx 941.92$ dollars.

Example 5.21. Chelsea's parents want to put money into a college fund so she can better afford the cost of college. Her family figures they can afford to save \$300 per month. They find that the Bank of America offers a nominal rate of 4.05% interest per year compounded monthly. How many years do they need to plan to invest the money to have \$40000 in the account when Chelsea goes to college?
Solution: We have $r = 0.0405$, $m = 12$, $R = 300$, $S = 40000$, and so:

$$40000 = \frac{300 \cdot ((1 + 0.0405/12)^{t \cdot 12} - 1)}{0.0405/12} = 88888.9 \cdot (1.00338^{12t} - 1)$$

and thus $t \approx 9.19$ years.

Proposition 5.22. *The present value of an annuity, called P, of n payments of R dollars each, paid at the end of each investment period into an account that earns interest at the rate of i per period, is:*

$$P = R\left(\frac{1-(1+i)^{-n}}{i}\right).$$

Using again that $i = r/m$ and $n = mt$, we also have:

Proposition 5.23. *The present value of an annuity, called P, of payments of R dollars each, paid at the end of each investment period into an account that earns a nominal annual rate of r compounded m times a year, for t years, is:*

$$P = R\left(\frac{1-\left(1+\frac{r}{m}\right)^{-mt}}{r/m}\right).$$

Example 5.24. An annuity is taken for 24 payments of \$100 each, made monthly and earning an annual interest of 9% per year, compounded monthly. What is the present value of the annuity?
Solution: We have $r = 0.09$, $m = 12$, $R = 100$, $t = 24/12 = 2$, and so:

$$P = \frac{100 \cdot (1-(1+0.09/12)^{-2\cdot 12})}{0.09/12} \approx 2188.91$$

is the present value of the annuity.

Example 5.25. A student owes \$6000 on a credit card that charges 21% annual interest compounded monthly on the unpaid balance. If the student makes monthly payments of \$120, how long will it take to pay it off? How much interest is paid in total? How much do you owe after 5 years?
Solution: We have $r = 0.21$, $m = 12$, $R = 120$, $P = 6000$, and so:

$$6000 = \frac{120 \cdot (1-(1+0.21/12)^{-t\cdot 12})}{0.21/12} = 6857.14 \cdot (1-1.0175^{-12t})$$

and thus it will take the student $t \approx 9.9885 \approx 10$ years. The student would have made $mt = 120$ payments of \$120 each, so it would have paid $120 \cdot 120 = 14400$ dollars, for a debt of 6000 dollars, and thus the student paid $14400 - 6000 = 8400$ dollars in interest.

Moreover, after 5 years, we have $r = 0.21$, $m = 12$, $R = 120$, $t = 5$, and so:

$$P = \frac{120 \cdot (1-(1+0.21/12)^{-5\cdot 12})}{0.21/12} \approx 4435.78$$

the student still owes \$4435.78 after 5 years of making payments.

Example 5.26. An employee announces to the company that employs him that they will retire in one year. The pension plan requires that the company pays the employee \$40000 in a lump sum at the end of one year, and every year thereafter until their death. The company makes the assumption that the employee will live to receive 25 payments. Interest rates are 5% per year compounded annually. How much money should the company set aside now to ensure that they can meet their pension obligations to the employee?
Solution: We have $r = 0.05$, $m = 1$, $t = 25$, $R = 40000$, and so:

$$P = \frac{40000 \cdot (1 - (1 + 0.05/1)^{-25 \cdot 1})}{0.05/1} \approx 563757.78$$

in dollars is how much the company should set aside.

Example 5.27. After making a down payment of \$2000 for a vehicle, Murphy paid \$200 per month for 36 months with an interest rate of 12% per year compounded monthly on the unpaid balance. What was the original cost of the car? How much did Murphy pay in interest charges?
Solution: We have $r = 0.12$, $m = 12$, $t = 36/12 = 3$ $R = 200$, and so the amount that Murphy paid after the down payment is:

$$P = \frac{200 \cdot (1 - (1 + 0.12/12)^{-12 \cdot 3})}{0.12/12} \approx 6021.5$$

dollars. Hence the original cost of the vehicle is $6021.5 + 2000 = 8021.5$. Since Murphy paid a total of $mtR = 12 \cdot 3 \cdot 200 = 7200$ the amount of interest he paid is $mtR - P = 12 \cdot 3 \cdot 200 - 6021.5 = 1178.5$ dollars.

5.3 Amortization and Sinking Funds

We can use the concept of present value of an annuity to analyze mortgages and financing of houses and cars, among others.

Example 5.28. A family wishes to purchase a house that costs \$130000. They make a down payment of \$20000 and finance the remainder for 30 years at 5.1% annual interest compounded monthly on the unpaid balance.

1. What are the monthly payments they need to make?
2. How much interest do they pay in total?
3. How much of the first payment is paid toward interest and how much is paid towards the current principal of the account?
4. How much do they still owe the bank after the first payment?
5. Suppose they instead finance the house for 15 years. What is the monthly payment? How much interest do they pay in total? How does the interest paid on a 15 year mortgage compare to a 30 year mortgage?

Solution: We have $r = 0.051$, $m = 12$, $t = 30$ $P = 130000 - 20000 = 110000$.

1. We have:
$$110000 = \frac{R \cdot (1 - (1 + 0.051/12)^{-30 \cdot 12})}{0.051/12} = 184.179R$$
and thus they need to make monthly payments of $R \approx 597.24$ dollars.

2. The total amount that they pay is $mtR = 12 \cdot 30 \cdot 597.24 = 215006.4$ and thus the interest they paid is $215006.4 - 110000 = 105006.4$ dollars.

3. Since they owe the bank \$110000 on the first payment, and that first payment the percentage of this that is interest is $0.051/12$, the interest they owe on the first payment is:
$$110000 \cdot \frac{0.051}{12} = 467.5.$$
Hence they pay 467.5 dollars towards the interest and $597.24 - 467.5 = 129.74$ dollars towards the principal, that is, the house.

4. They still owe the bank $110000 - 129.74 = 109870.26$ dollars after the first payment.

   Another way to see these last two points is that the total they owe to the bank once counted the interest is $110000 + 467.5 = 110467.5$ dollars. Since they make payments of 597.24 dollars, what remains after the first payment is $110467.5 - 597.24 = 109870.26$ dollars. Hence they only contributed $110000 - 109870.26 = 129.74$ dollars towards the principal, and they contributed $597.24 - 129.74 = 467.5$ dollars towards the interest.

5. It they instead finance the house for 15 years, we have $r = 0.051$, $m = 12$, $t = 15$ $P = 130000 - 20000 = 110000$. The monthly payment is then:
$$110000 = \frac{R \cdot (1 - (1 + 0.051/12)^{-15 \cdot 12})}{0.051/12} = 125.626R$$
and thus they need to make monthly payments of $R \approx 875.61$ dollars. They pay a total of $mtR = 12 \cdot 15 \cdot 875.61 = 157609.8$ and thus the interest they paid is $157609.8 - 110000 = 47609.8$ dollars. This means that the interest paid on a 30 year mortgage is $105006.4 - 47609.8 = 57396.6$ dollars more than the interest paid on a 15 year mortgage.

Example 5.29. A student wishes to borrow \$20000 from a bank to purchase a car. The bank charges an annual interest rate of 4.2% compounded monthly, and the student finances the loan for 5 years. How much must the student pay per month so that the loan will be paid off after 5 years? How much must the student pay per month so that the loan will be paid off after 4 years? How much must the student pay per month so that the loan will be paid off after 6 years? Find the total interest paid on the loan for each of the different loan periods.
Solution: We have $r = 0.042$, $m = 12$, $P = 20000$. We then have that:

1. if $t = 5$ then:

$$20000 = \frac{R \cdot (1 - (1 + 0.042/12)^{-12 \cdot 5})}{0.042/12} = 54.0339R$$

   and thus the student needs to make monthly payments of $R \approx 370.14$ dollars. The interest paid in dollars is:

$$mtR - P = 12 \cdot 5 \cdot 370.14 - 20000 = 2208.4.$$

2. if $t = 4$ then:

$$20000 = \frac{R \cdot (1 - (1 + 0.042/12)^{-12 \cdot 4})}{0.042/12} = 44.1138R$$

   and thus the student needs to make monthly payments of $R \approx 453.37$ dollars. The interest paid in dollars is:

$$mtR - P = 12 \cdot 4 \cdot 453.37 - 20000 = 1761.76.$$

3. if $t = 6$ then:

$$20000 = \frac{R \cdot (1 - (1 + 0.042/12)^{-12 \cdot 6})}{0.042/12} = 63.5466R$$

   and thus the student needs to make monthly payments of $R \approx 314.73$ dollars. The interest paid in dollars is:

$$mtR - P = 12 \cdot 6 \cdot 314.73 - 20000 = 2660.56.$$

Definition 5.30. *In the context of a loan being amortized, an* amortization table *is a table showing for each payment period how much is being paid, how much of each payment is interest, how much of each payment is paid towards the principal, and how much is the outstanding principal.*

Example 5.31. A student borrows \$3000 from a bank at an annual interest rate of 12% compounded quarterly for 2 years. Find the monthly payment and then complete a table showing for each quarter much is being paid, how much of each payment is interest, how much of each payment is paid towards the principal, and how much is the outstanding principal. That is, find an amortization table of the transaction.
Solution: We have $r = 0.12$, $m = 4$, $t = 2$, $P = 3000$. We then have that:

$$3000 = \frac{R \cdot (1 - (1 + 0.12/4)^{-4 \cdot 2})}{0.12/4} = 7.01969R$$

and thus the student needs to make quarterly payments of $R \approx 427.37$ dollars. An amortization table in dollars is:

Period	Payment	Interest	Toward principal	Outstanding principal
0				3000
1	427.37	90	337.37	2662.63
2	427.37	79.88	347.49	2315.14
3	427.37	69.45	357.92	1957.22
4	427.37	58.72	368.65	1588.57
5	427.37	47.66	379.71	1208.86
6	427.37	36.27	391.1	817.76
7	427.37	24.53	402.84	414.92
8	427.37	12.45	414.92	0

Definition 5.32. *In the context of an annuity, the* equity *that the taker of the annuity has is the amount, including interest, that they have accumulated with their deposits. In the context of a loan being amortized, the equity that the taker of the loan has is the amount that they have contributed to the payment of the original principal.*

Example 5.33. A student wishes to purchase a house that costs \$130000. For this, they make a down payment of \$20000, and finance the remainder \$110000 for 30 years at 5.1% annual interest compounded monthly. How much is the monthly payment on this mortgage? Find an amortization table. How much equity will the student have after 10 years? How much equity will the student have after 10 years?
Solution: We have $r = 0.051$, $m = 12$, $t = 30$, $P = 110000$, and thus we have:

$$110000 = \frac{R \cdot (1 - (1 + 0.051/12)^{-30 \cdot 12})}{0.051/12} = 184.179R$$

and thus the student needs to make monthly payments of $R \approx 597.24$ dollars. We will use the abbreviations PMT for payment, $PMTL$ for the number of payments left, I for interest, TP for the amount going toward the principal, OP for the outstanding principal, and E for the equity. The first 7 rows of an amortization table in dollars are then:

Period	$PMTL$	PMT	I	TP	OP	E
0	360				110000	20000
1	359	597.24	467.5	129.74	109870.26	20129.74
2	358	597.24	466.95	130.29	109739.97	20260.03
3	357	597.24	466.39	130.85	109609.12	20390.88
4	356	597.24	465.84	131.4	109477.12	20522.28
5	355	597.24	463.28	131.96	109345.76	20654.24
6	354	597.24	464.72	132.52	109213.24	20786.76

To compute the equity after 10 years, we can compute how much outstanding payment remains (that is, how much do we still owe the bank) after 10 years. Since we still have 20 years till we pay off the loan, the outstanding payment is the present value of an annuity with $r = 0.051$, $m = 12$, $t = 20$, $R = 597.24$, that is:

$$P = \frac{597.24 \cdot (1 - (1 + 0.051/12)^{-20 \cdot 12})}{0.051/12} \approx 89744.07$$

dollars. This means that we have paid $110000 - 89744.07 = 20255.93$ dollars from the loan. Hence our equity, which is how much have we paid (including the down payment), is $20255.93 + 20000 = 40255.93$ dollars.

To compute the equity after 18 years, we can compute how much outstanding payment remains (that is, how much do we still owe the bank) after 18 years. Since we still have 12 years till we pay off the loan, the outstanding payment is the present value of an annuity with $r = 0.051$, $m = 12$, $t = 12$, $R = 597.24$, that is:

$$P = \frac{597.24 \cdot (1 - (1 + 0.051/12)^{-12 \cdot 12})}{0.051/12} \approx 64225.23$$

dollars. This means that we have paid $110000 - 64225.23 = 45774.77$ dollars from the loan. Hence our equity, which is how much have we paid (including the down payment), is $45774.77 + 20000 = 65774.77$ dollars.

Definition 5.34. *A* sinking fund *is an account that is set up for a specific purpose at some future date.*

Example 5.35. The owner of Zara has decided to set up a sinking fund for the purpose of expanding in Latin America in two years' time. It is expected that the expansion will cost 30000 million dollars. If the fund earns 10% interest per year, compounded quarterly, determine the size of each (equal) quarterly installment that Amancio Ortega (Zara's owner) should pay into the fund. Verify the result by computing a table showing for each payment period how much is being deposited, how much interest the fund is earning, how much it is being added to the fund, and how much is being accumulated in the fund.

Solution: We have $r = 0.10$, $m = 4$, $t = 2$, $S = 30000$. We then have that:

$$30000 = \frac{R \cdot ((1 + 0.10/4)^{4 \cdot 2} - 1)}{0.10/4} = 8.73612R$$

and thus Amancio Ortega needs to make quarterly payments of $R \approx 3434.02$ million dollars. The desired table in millions of dollars is:

Period	Deposited	Interest earned	Addition to fund	Accumulated in fund
1	3434.02		3434.02	3434.02
2	3434.02	85.85	3519.87	6953.89
3	3434.02	173.85	3607.87	10561.76
4	3434.02	264.04	3698.06	14259.82
5	3434.02	356.50	3790.52	18050.34
6	3434.02	451.26	3885.28	21935.62
7	3434.02	548.39	3982.41	25918.03
8	3434.02	697.95	4081.97	30000

Example 5.36. The Umbrella Corporation wishes to set up a sinking fund to replace a current laboratory machine. It is estimated that the machine will need to be replaced in 10 years and will cost 100000 million dollars. How much per quarter should be deposited into an account with an annual interest rate of 8%, compounded quarterly, to meet this future obligation? What will be the total amount of the payments? How much will be the interest earned? If only four years after making the sinking fund the Umbrella Corporation decides to use the accumulated money (that is, their equity) for another purpose, determine how much money they have available.
Solution: We have $r = 0.08$, $m = 4$, $t = 10$, $S = 100000$. We then have that:

$$100000 = \frac{R \cdot ((1 + 0.08/4)^{4 \cdot 10} - 1)}{0.08/4} = 60.402R$$

and thus the Umbrella Corporation needs to make quarterly payments of $R \approx 1655.57$ million dollars. They deposit a total of $mtR = 4 \cdot 10 \cdot 1655.57 = 66222.8$ million dollars, meaning that they earned $100000 - 66222.8 = 33777.2$ million dollars from interest.

The equity will be given by the final accumulated value of an annuity with $r = 0.08$, $m = 4$, $t = 4$, $R = 1655.57$. Hence they have:

$$S = \frac{1655.57 \cdot ((1 + 0.08/4)^{4 \cdot 4} - 1)}{0.08/4} \approx 30858.64$$

million dollars available.

6 Sets and Counting

6.1 Sets and Set Operations

In daily life we often deal with collections of objects, such as our belongings, the appointments in our schedule, or the available gas stations at a given time of the day.

Definition 6.1. *A* set *is a well defined unordered collection of objects. Namely, given a set and any object, we must be able to determine whether or not the object belongs to the set. The objects forming a set are called the* elements of the set.

There are two main ways of presenting a set. The *roster notation* gives a set by listing all of its elements. The *set-builder notation* gives a set by describing the property or properties that an object must satisfy to be in the set.

Example 6.2. Give the roster notation and the set-builder notation of the set formed by all the letters of the English alphabet.
Solution: The roster notation is:

$$A = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}.$$

The set-builder notation is:

$$A = \{x | x \text{ is a letter of the English alphabet}\}.$$

Definition 6.3. *Consider a set A and an element a. If a is an element of the set A, we write $a \in A$ and say that a* belongs to A *or* a is in A. *If a is not an element of the set A, we write $a \notin A$ and say that a* does not belongs to A *or* a is not in A.

Example 6.4. Consider $A = \{1, 2, 3, 4, 5\}$. Name an element that belongs to A and an element that does not belong to A.
Solution: We have that $3 \in A$, and also $6 \notin A$.

Definition 6.5. *Consider two sets A and B. We say that they are* equal*, and we write $A = B$, whenever they have exactly the same elements.*

Example 6.6. Consider:

$$\begin{aligned} A &= \{a, e, i, o, u\}, \\ B &= \{u, o, i, e, a\}, \\ C &= \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}. \end{aligned}$$

Determine which sets are equal, which are different, and why.
Solution: We have that $A = B$ because every element of A is in B, and every element of B is in A. Since we have $a \in A$, $a \in B$, but $a \notin C$, we have $A \neq C$ and $B \neq C$.

Definition 6.7. *Consider two sets A and B. We say that A is a* subset *of B, and we write $A \subseteq B$, whenever every element of A is also an element of B. If A is a strict subset of B, that is if A is a subset of B that is not equal to B, we write $A \subsetneq B$ and we say that A is a* proper subset *of B. If A is not a subset of B, we write $A \nsubseteq B$.*

Example 6.8. Consider:

$$\begin{aligned}
A &= \{a, e, i, o, u\},\\
B &= \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\},\\
C &= \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}.
\end{aligned}$$

Determine which sets are equal, which are different, what sets are included in others, and why.

Solution: We have $a \in A$ but $a \notin B$, so $A \neq B$ and $A \nsubseteq B$. We also have $b \in B$ but $b \notin A$, so we have $B \nsubseteq A$. We have that every element in A is also in C, but $b \in C$ while $b \notin A$, so $A \neq C$ and $A \subsetneq C$ and $C \nsubseteq A$. We have that every element in B is also in C, but $a \in C$ while $a \notin B$, so $B \neq C$ and $B \subsetneq C$ and $C \nsubseteq B$.

Proposition 6.9. *Let A and B be two sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

Definition 6.10. *The set $\{\}$ containing no elements is called the* empty set*, and we denote it by $\emptyset$.*

Example 6.11. List all subsets of $A = \{1, 2, 3\}$.

Solution: We can determine the subsets in terms of how many elements they have. There is one subset of A containing no elements: the empty set $\emptyset$. There are three sets containing one element: $\{1\}$, $\{2\}$, and $\{3\}$. There are three subsets containing two elements: $\{1, 2\}$, $\{2, 3\}$, and $\{1, 3\}$. There is one subset containing three elements: $\{1, 2, 3\}$. Therefore the subsets of A are:

$$\emptyset, \quad \{1\}, \quad \{2\}, \quad \{3\}, \quad \{1, 2\}, \quad \{2, 3\}, \quad \{1, 3\}, \quad \{1, 2, 3\}.$$

Definition 6.12. *The* universal set *of a particular setting is the set containing all the elements of interest in that particular setting. We denote it U.*

Example 6.13. In the setting of determining the ratio of female to male students at Texas A&M University, what is the universal set?

Solution: Since the elements of interest are the female and male students of the University, the universal set is the set consisting of all female and male students of the University.

Example 6.14. In the setting of determining the ratio of female to male students within Mays Business School, what is the universal set?
Solution: Since the elements of interest are the female and male students within Mays Business School, the universal set is the set consisting of all female and male students within Mays Business School.

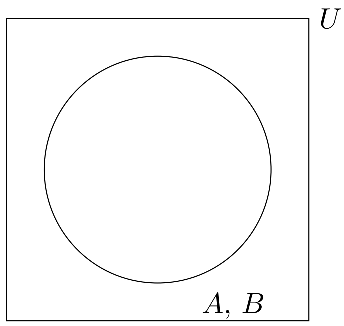
Definition 6.15. *A* Venn diagram *is a visual representation of sets. We represent the universal set U by a rectangle, containing regions which are the other sets of interest.*

Example 6.16. Use Venn diagrams to illustrate:

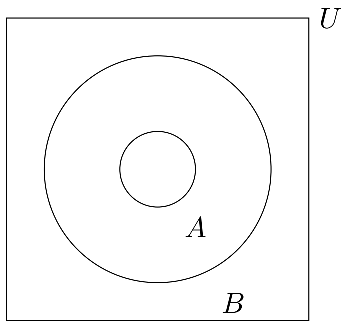
1. The sets A and B are equal.
2. The set A is a proper subset of the set B.
3. The sets A and B are not subsets of each other.

Solution:

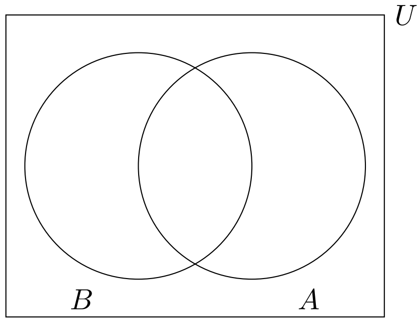
1. For $A = B$ within U we have:



2. For $A \subsetneq B$ within U we have:



3. For $A \neq B$ within U we have:



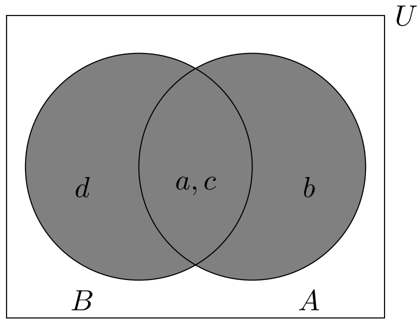
Given any number of sets, there are certain operations that we can perform. This is simply a combination of sets to yield another set. We will assume that all such sets under which we are performing operations are subsets of a global universal set U.

Definition 6.17. *Let A and B be two sets. The* union *of A and B, written $A \cup B$, is the set of all elements that belong to either A or B or both:*

$$A \cup B = \{x | x \in A, \text{ or } x \in B, \text{ or both}\}.$$

Example 6.18. What is the union of $A = \{a, b, c\}$ and $B = \{a, c, d\}$?
Solution: We need to include all the elements that are just in A, that is b, the elements that are just in B, that is d, and the elements that are both in A and B, that is a and c. We then have $A \cup B = \{a, b, c, d\}$.

This can also be visualized with a Venn diagram:



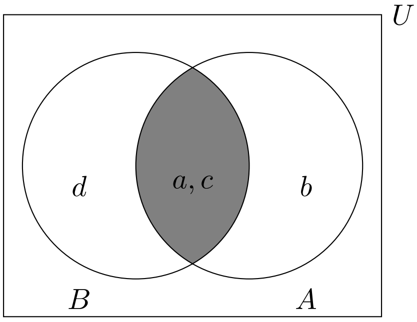
Definition 6.19. *Let A and B be two sets. The* intersection *of A and B, written $A \cap B$, is the set of all elements that belong to both A and B:*

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

If two sets have no elements in common, they are said to be disjoint.

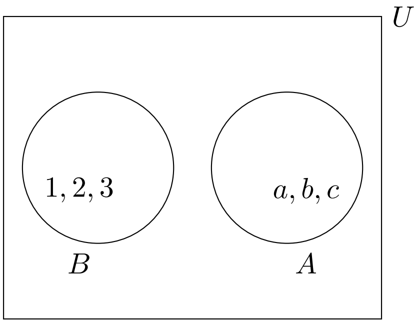
Example 6.20. What is the intersection of $A = \{a, b, c\}$ and $B = \{a, c, d\}$?
Solution: We need to include all the elements that are in both A and B, that is a and c. We then have $A \cap B = \{a, c\}$.

This can also be visualized with a Venn diagram:



Example 6.21. What is the intersection of $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$?
Solution: We need to include all the elements that are in both A and B. However, A and B have no elements in common, they are disjoint. We then have $A \cap B = \{\} = \emptyset$.

This can also be visualized with a Venn diagram:

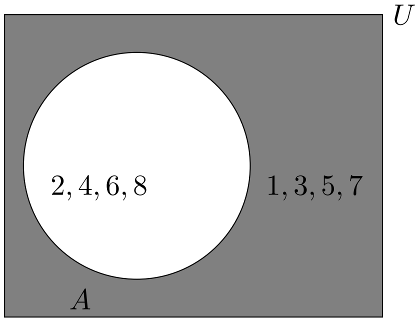


Definition 6.22. *Let A be a sets. The* complement *of A, written A^c, is the set of all elements in U that do not belong to A:*

$$A^c = \{x | x \in U \text{ and } x \notin A\}.$$

Example 6.23. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A = \{2, 4, 6, 8\}$. What is the complement of A?
Solution: We need to include all the elements that are in U bot not in A. Then $A^c = \{1, 3, 5, 7\}$.

This can also be visualized with a Venn diagram:



Proposition 6.24. *Given A, B, C three sets, we have that unions and intersections are commutative:*

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Proposition 6.25. *Given A, B two sets, we have that unions and intersections are associative:*

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Proposition 6.26. *Given A, B, C three sets, we have that unions and intersections are distributive with respect to each other:*

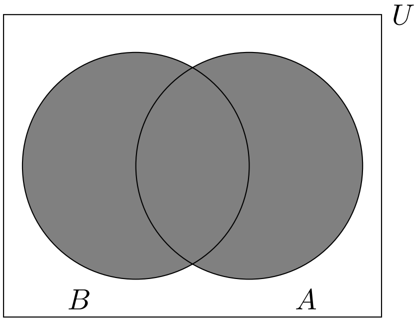
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Theorem 6.27. *Given A, B two sets, we have:*

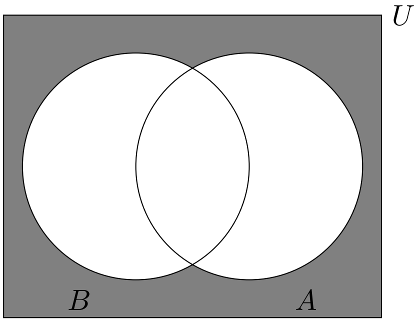
$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

which are called De Morgan's Laws*.*

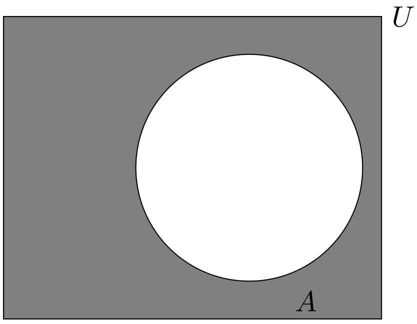
Example 6.28. Using Venn diagrams, show that $(A \cup B)^c = A^c \cup B^c$.
Solution: The general Venn diagram of $A \cup B$ is:



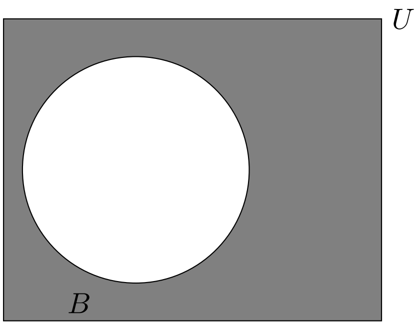
so the general Venn diagram of $(A \cup B)^c$ is:



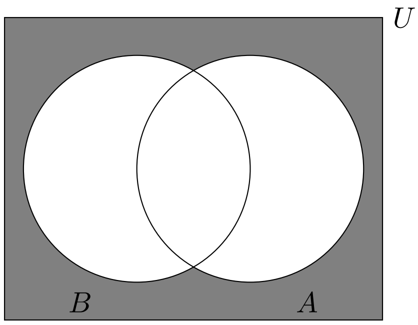
The general Venn diagram of A^c is



and the general Venn diagram of B^c is:



so the general Venn diagram of $A^c \cap B^c$ is their intersection:



which is exactly the same diagram obtained before.

Example 6.29. Let $U = \{a, e, i, o, u, v, w, x, y, z\}$, $A = \{a, e, i, o, u\}$, $B = \{u, v, w, x\}$. Verify by direct computation that $(A \cup B)^c = A^c \cap B^c$.
Solution: We have $A \cup B = \{a, e, i, o, u, v, w, x\}$, so $(A \cup B)^c = \{y, z\}$. We have $A^c = \{v, w, x, y, z\}$ and $B^c = \{a, e, i, o, u, y, z\}$, so $A^c \cap B^c = \{y, z\}$.

Example 6.30. Let U denote the set of all cars in Allen Hoda's dealership. Consider the following sets:

1. $A = \{x \in U | x \text{ has automatic transmission}\}$,
2. $B = \{x \in U | x \text{ has air conditioning}\}$,
3. $C = \{x \in U | x \text{ has sunroof}\}$.

Find an expression in terms of A, B, and C for each of the following sets:

1. The set of cars with at least one of the given options.
2. The set of cars with exactly one of the given options.
3. The set of cars with automatic transmission and sunroof but no air conditioning.

Solution: We have:

1. $A \cup B \cup C$.
2. $(A \cap B^c \cap C^c) \cup (B \cap A^c \cap B^c) \cup (C \cap A^c \cap B^c)$.
3. $A \cap C \cap B^c$.

6.2 The Number of Elements in a Finite Set

Knowing how many elements a set has is useful and desirable.

Definition 6.31. *Let A be a set. We say that A is a* finite set *if it has finitely many elements. We denote by $n(A)$ the number of elements in A, and we call it the* cardinality of A.

Example 6.32. Determine the number of elements of the following sets:

$$\begin{aligned}
A &= \{x|x \text{ is a letter of the English alphabet}\},\\
B &= \{x|x \text{ is a state of the United States of America}\},\\
C &= \{x|x \text{ is a college of Texas A\&M University}\}.
\end{aligned}$$

Solution: We have $n(A) = 26$, $n(B) = 50$, and $n(C) = 18$.

Example 6.33. Let $A = \{a, b, c, d, e\}$, $B = \{d, e, f, g, h\}$. Determine the number of elements in their union.
Solution: We have $n(A \cup B) = 8$.

Proposition 6.34. *Let A_1, A_2 be two finite sets, then:*

$$n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2).$$

Proposition 6.35 (Counting Principle)**.** *Let $A_1, \dots, A_n$ be finite sets, then:*

$$n(A_1 \cup \cdots \cup A_n) = \sum_{j=1}^{n} (-1)^{j-1} \sum_{1 \le i_1 < \cdots < i_j \le n} n(A_{i_1} \cap \cdots \cap A_{i_j})$$

The case $n = 2$ above is key in understanding the counting principle. Another specific case that is very useful is the case $n = 3$:

$$\begin{aligned}
n(A_1 \cup A_2 \cup A_3) = {} & n(A_1) + n(A_2) + n(A_3)\\
& - n(A_1 \cap A_2) - n(A_1 \cap A_3) - n(A_2 \cap A_3) + n(A_1 \cap A_2 \cap A_3).
\end{aligned}$$

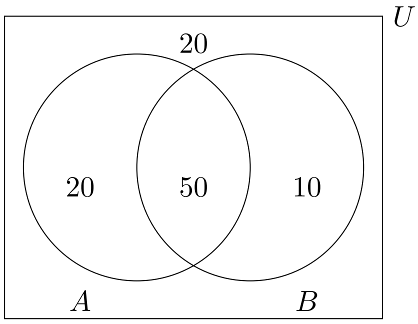
Example 6.36. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8\}$. Verify the counting principle.
Solution: We have:

$$n(A \cup B \cup C) = 8$$

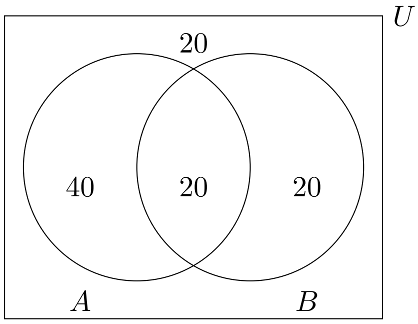
and since $n(A) = 4$, $n(B) = 5$, $n(C) = 5$, $n(A \cap B) = 3$, $n(A \cap C) = 1$, $n(B \cap C) = 3$, $n(A \cap B \cap C) = 1$ we have:

$$\begin{aligned}
& n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) - n(A_1 \cap A_3) - n(A_2 \cap A_3)\\
& \qquad + n(A_1 \cap A_2 \cap A_3) = 4 + 5 + 5 - 3 - 1 - 3 + 1 = 8.
\end{aligned}$$

Example 6.37. The Starbucks at Evans Library surveyed 100 customers. They found that 70 customers use sugar, 60 customers use cream, and 50 use both sugar and cream. How many consumers surveyed took sugar or cream? Make a Venn diagram conveying the situation.
Solution: Let A and B be the consumers using sugar and cream, respectively. We have that $n(A \cup B) = 80$ consumers take sugar or cream.



Example 6.38. Let A and B be subsets of a universal set U. Suppose that $n(U) = 100$, $n(A) = 60$, $n(B) = 40$, and $n(A \cap B) = 20$. Compute $n(A \cup B)$, $n(A \cap B^c)$, $n(A^c \cap B)$. Make a Venn diagram conveying the situation.
Solution: We have that $n(A \cup B) = 80$, $n(A \cap B^c) = 40$, $n(A^c \cap B) = 20$.

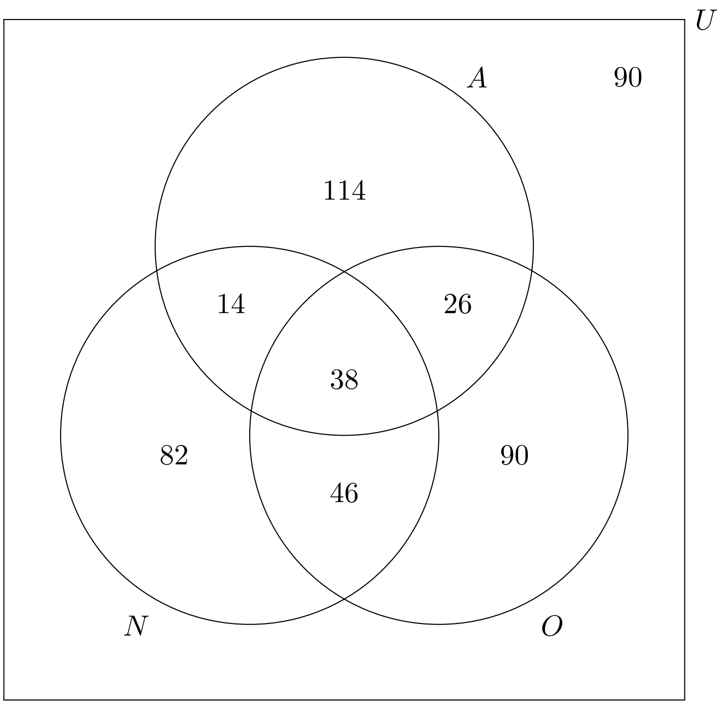


Example 6.39. The company Johnson & Johnson advertises their products in three magazines: The New Yorker, The Old Yorker, and The Ancient Yorker. A survey of 500 customers revealed that:

1. 180 knew their products from The New Yorker,
2. 200 knew their products from The Old Yorker,
3. 192 knew their products from The Ancient Yorker,
4. 84 knew their products from The New Yorker and The Old Yorker,
5. 52 knew their products from The New Yorker and The Ancient Yorker,
6. 64 knew their products from The Old Yorker and The Ancient Yorker,

7. 38 knew their products from The New Yorker, The Old Yorker, and The Ancient Yorker.

How many of the customers saw the manufacturer's advertisement in at least one magazine? How many of the customers saw the manufacturer's advertisement in exactly one magazine? How many of the customers saw the manufacturer's advertisement in at least two magazines? How many of the customers saw the manufacturer's advertisement in exactly two magazines? Make a Venn diagram conveying the situation.
Solution: Let N, O, and A be the customers knowing the products from The New Yorker, The Old Yorker, and The Ancient Yorker respectively. The customers who know the products from at least one magazine are $n(N \cup O \cup A) = 410$. The customers who know the products from exactly one magazine are $n(N \cap O^c \cap A^c) + n(N^c \cap O \cap A^c) + n(N^c \cap O^c \cap A) = 286$. The customers who know the products from at least two magazines are $n(N \cap O \cap A^c) + n(N \cap O^c \cap A) + n(N^c \cap O \cap A) + n(N \cap O \cap A) = 124$. The customers who know the products from exactly two magazines are $n(N \cap O \cap A^c) + n(N \cap O^c \cap A) + n(N^c \cap O \cap A) + n(N \cap O \cap A) = 86$.

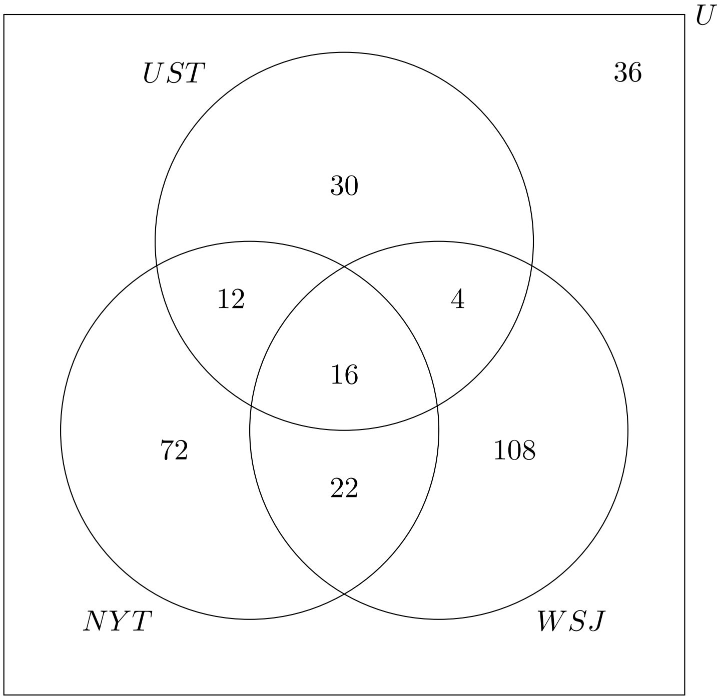


Example 6.40. In a survey of 300 investors regarding subscriptions to the New York Times, the Wall Street Journal, and USA Today, the following were found:

1. 122 subscribe to the NYT,
2. 150 subscribe to the WSJ,
3. 62 subscribe to the UST,

4. 38 subscribe to the NYT and the WSJ,
5. 28 subscribe to the NYT and the UST,
6. 20 subscribe to the WSJ and the UST,
7. 36 do not subscribe to any of these three newspapers.

How many of the investors surveyed subscribe to all three newspapers? How many of the investors subscribe to only one of these newspapers? Make a Venn diagram conveying the situation.
Solution: There are 16 surveyed investors subscribed to all three newspapers. There are 210 surveyed investors subscribed to only one newspaper.



6.3 The Multiplication Principle

There are other tools that we can use when determining the cardinality of a set.

Proposition 6.41 (Multiplication Principle). *If a set $S = \{s_1, \ldots, s_n\}$ can be constructed in such a way that for each $1 \leq i \leq n$ there are m_i choices for s_i, then the number of distinct sets which can be constructed is $m_1 \cdots m_n$.*

Example 6.42. The dealership Allen Honda needs to transfer cars to Houston. For this they employ an eighteen wheeler that can hold three vehicles at a time, but the vehicles cannot be of the same color. Determine the number of distinct shipments

possible knowing that there are seven red cars, five green cards, and nine blue cars on the dealership that need to be transferred.
Solution: There are 315 possible shipments.

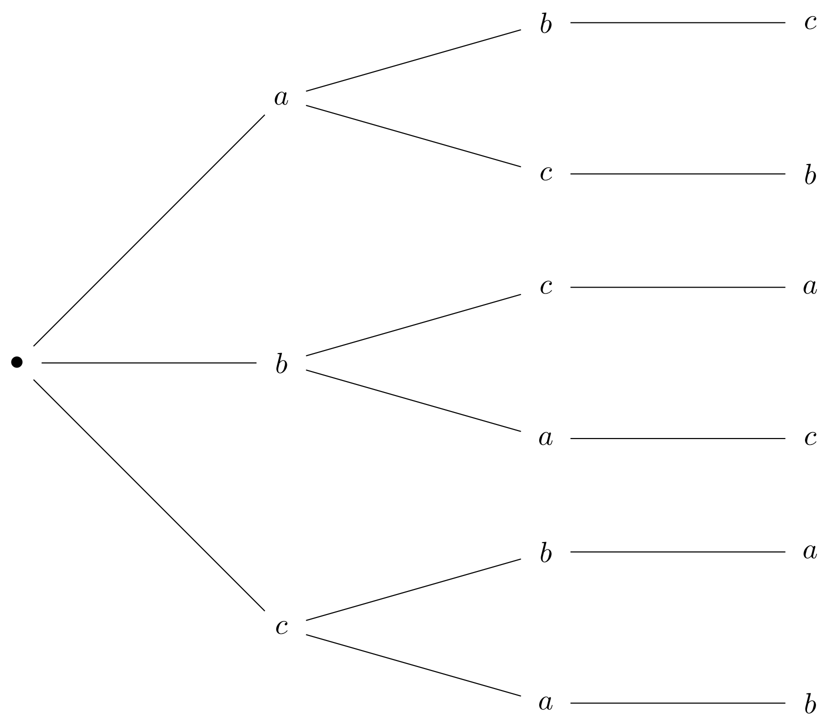
Example 6.43. A coin is tossed three times, and the sequence of heads and tails is recorded. What is the possible number of outcomes of the sequences? You may use a tree diagram.
Solution: There are 8 possible sequences.

6.4 Permutations and Combinations

We will now apply the multiplication principle to solve counting problems. We start by considering setups where the order of the elements matters.

Definition 6.44. *Let S be a set containing distinct objects, a* permutation *of the set is an arrangement of these objects in a concrete order.*

Example 6.45. Let $A = \{a, b, c\}$. Find the number of permutations of A. List all permutations of A using a tree diagram.
Solution: There are 6 permutations. We can list them as:



Example 6.46. Find the number of ways the Texas A&M starting baseball team (consisting of nine players) can arrange themselves in a line for a group picture.
Solution: There are 362880 ways.

Definition 6.47. *Given $n \in \mathbb{N}$, we denote by $n!$ the* factorial of n. *We set $0! = 1$ and for $n > 0$:*

$$n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1.$$

Proposition 6.48. *If we take n elements from n elements, the possible number of permutations is $n!$.*

Proposition 6.49. *If we take r elements from n elements, the possible number of permutations is:*

$$\frac{n!}{(n-r)!}.$$

Definition 6.50. *Given $n, r \in \mathbb{N}$ with $r \leq n$, we call the* permutations of n objects taken r at a time *to the number:*

$$P(n,r) = \frac{n!}{(n-r)!}.$$

Example 6.51. Find the number of ways the Texas A&M Student Senate can choose a chairman, a vice-president, a secretary, and a treasurer, if there are 120 eligible members.
Solution: There are 197149680 ways.

Theorem 6.52. *Given a set of n objects, suppose where we have r different types of objects, and of each type we have n_i, $1 \leq i \leq r$ many, so that $n_1 + \dots + n_r = n$. Then the number of permutations of these n objects taken n at a time is:*

$$\frac{n!}{n_1! \cdots n_r!}.$$

Example 6.53. Find the number of anagrams that can be formed from the word MISSISSIPPI.
Solution: There are 34650 anagrams.

Example 6.54. The Standard in College Station received 12 resident complaints last month. In how many ways can the complaints be addressed by four of the Standard's lawyers if each lawyer handles three inquiries?
Solution: There are:

$$\frac{12!}{3! \cdot 3! \cdot 3! \cdot 3!} = 369600$$

ways of addressing them.

So far we have been interested in arrangements where the order of the elements matters. However, in many situations the order does not matter.

Definition 6.55. *Let S be a set containing distinct objects, a* combination *of the set is a selection of these objects without regarding their order.*

Proposition 6.56. *If we take r elements from n elements, the possible number of combinations is:*

$$\frac{n!}{r!(n-r)!}.$$

Example 6.57. Calculate how many different four digit integers larger than 3000 can be made from the numbers $\{1, 1, 3, 5, 8\}$.
Solution: It is helpful to think about the positions: we can choose among $\{8, 3, 5\}$ for the thousands, we can choose among 4 options for the hundreds (any of the not chosen for thousands), we can choose among 3 options for the tens (any of the not chosen for thousands nor hundreds), and we can choose among 2 options for the ones (any of the not chosen for thousands nor hundreds nor tens). We can then make 72 different integers larger than 3000.

Definition 6.58. *Given $n, r \in \mathbb{N}$ with $r \leq n$, we call the* combinations of n objects taken r at a time *to the number:*

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

Example 6.59. A container transporting cars from Allen Honda to Houston holds six aligned distinct Civic: one red, one green, and four blue. If the blue cars must be stored next to one another, and the green car must be by the container door, determine the number of storage arrangements.
Solution: There are 48 arrangements.

Example 6.60. The Houston Symphonic String Quartet is composed of two violinists, a violist, and a cellist. The eligible members of the Houston Symphony are six violinists, three violists, and two cellists.

1. In how many ways can the string quartet be formed?
2. In how many ways can the string quartet be formed if one of the violinists is to be designated as the first violinist, and the other is to be designated as the second violinist?

Solution:

1. Since here the order doesn't matter, there are $C(6,2)C(3,1)C(2,1) = 90$ ways.

2. Since here the order of the violinists matter, there are $P(6,2)C(3,1)C(2,1) = 180$ ways.

Example 6.61. The heavy metal band Gloryhammer are planning a Texas tour with seven performances in Houston, Dallas, Austin, San Antonio, Waco, Round Rock, and El Paso. In how many ways can they arrange their itinerary if:

1. There are no restrictions.

2. The three performances of Central Texas must be given consecutively.

Solution:

1. Since the order matters, there are $P(7,7) = 5040$ ways.

2. Treating the performances of Central Texas as a unit, we have $P(5,5)$ ways of arranging the performances. Now we have $P(3,3)$ ways of arranging the performances in Central Texas. Then we have $P(5,5)P(3,3) = 720$ ways.

Example 6.62. The U.N. Security Council consists of 5 permanent members (China, France, Russia, United Kingdom, United States) and 10 non-permanent members. Any measure requires nine votes to pass. The permanent members have veto power, that is, by voting against any decision they automatically block it. In how many ways can a measure be passed if there are no abstentions?
Solution: Since the permanent members need to vote for the measure, this can be done in $C(5,5) = 1$ way. For the decision to pass, at least 4 of the other members must also vote for the measure. Thus there are:

$$C(5,5)(C(10,4)+C(10,5)+C(10,6)+C(10,7)+C(10,8)+C(10,9)+C(10,10)) = 848$$

ways in which the measure can pass.

7 Probability

We will use probability to tackle problems dealing with issues where an element of uncertainty or chance is present.

7.1 Experiments, Sample Spaces, and Events

Definition 7.1. *An* experiment *is an activity with observable results. The results of the experiment are called* outcomes*.*

Tossing a coin and observing on which side it falls, rolling a dice and observing which of the sides shows up, testing a product from a batch and observing whether it is defective, are all experiments with possible outcomes.

Definition 7.2. *Consider an experiment. We call* sample point *a possible outcome of the experiment. We call* sample space *the set consisting of all possible outcomes of the experiment. We call* event *to a subset of a sample space of the experiment. An event is comprised of elements satisfying a specific experimental condition.*

Example 7.3. A pair of six sided die are rolled together, and the face up values are recorded. Determine what the sample space of this experiment is. Determine the events E_i that the sum of the face up values is i, for $2 \leq i \leq 12$. Describe the event of the first dice having a value of 6 as an intersection of two other events.
Solution: The sample space is given by the entries of the following table:

	1	2	3	4	5	6
1	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
2	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
3	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$
4	$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$
5	$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$
6	$(6,1)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$

and thus the following table determines the events E_i for $2 \leq i \leq 12$:

Sum	Event
2	$E_2 = \{(1,1)\}$
3	$E_3 = \{(1,2),(2,1)\}$
4	$E_4 = \{(1,3),(2,2),(3,1)\}$
5	$E_5 = \{(1,4),(2,3),(3,2),(4,1)\}$
6	$E_6 = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$
7	$E_7 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$
8	$E_8 = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$
9	$E_9 = \{(3,6),(4,5),(5,4),(6,3)\}$
10	$E_{10} = \{(4,6),(5,5),(6,4)\}$
11	$E_{11} = \{(5,6),(6,5)\}$
12	$E_{12} = \{(6,6)\}$

and we can describe the event of the first dice having a value of 6 as the intersection of the events:

1. A_1 the event of the first dice having an even value.
2. A_2 the event of the first dice having a value divisible by three.

Definition 7.4. *Let S be the sample space of an experiment. A collection A_1, A_2, ..., A_n of events is called* mutually exclusive *if $A_i \cap A_j = \emptyset$ whenever $i \neq j$. A collection $A_1, A_2, \ldots, A_n$ of events is called* mutually exhaustive *if $A_i \cup A_j = S$ whenever $i \neq j$.*

A pair of six sided die are rolled together, and S denotes the sample space of all possible pairs of face up values. Provide examples of mutually exclusive events, and examples of mutually exhaustive events.

Example 7.5. A pair of six sided die are rolled together, and the face up values are recorded. Provide examples of mutually exclusive events. Provide examples of mutually exhaustive events.
Solution: The events:

1. A_1 the event of the first dice having an even value.
2. A_2 the event of the first dice having an odd value.

are mutually exclusive. The events:

1. A_1 the event of the first dice having an value equal or smaller than 4.
2. A_2 the event of the values of both die adding up to 5 or above.

are mutually exhaustive.

7.2 Definition of Probability

The concept of probability comes from generalizing the expected outcome of experiments. We thus have two different ideas of probability: the experimental one and the theoretical one.

Definition 7.6. *Consider an experiment repeated n times. The event E occurs m times. We call the* relative frequency *of E to the ratio m/n. If the relative frequency approaches some value when we keep repeating the experiment, we call this value $P(E)$ the* empirical probability*.*

In this sense, the empirical probability $P(E)$ of an event occurring is a measure of the proportion of the time that the event E will occur in the long run.

Definition 7.7. *Given a sample space $\mathcal{S}$ of an experiment and E an event of the experiment, a* probability *is a function $P : \mathcal{S} \longrightarrow [0,1]$ satisfying $P(\emptyset) = 0$ and $P(\mathcal{S}) = 1$. The probability of an event is the sum of the probabilities of all sample points making up the event.*

In this sense, the probability $P(E)$ of an event occurring measures the likelihood of the event E occurring. Each sample point (that is, outcome) in a sample space (coming from an experiment) is assigned a probability (that lies between zero and one), and each event is assigned a probability.

Proposition 7.8. *Given a sample space $\mathcal{S}$, an event E consisting of the sample points $s_1, \ldots, s_n$, and a probability $P : \mathcal{S} \longrightarrow [0,1]$, then $P(E) = P(s_1) + \cdots + P(s_n)$.*

As a consequence, we obtain the important result:

Theorem 7.9. *Given a finite sample space $\mathcal{S}$ consisting of the sample points $s_1, \ldots, s_n$, and a probability $P : \mathcal{S} \longrightarrow [0,1]$, then $P(S) = P(s_1) + \cdots + P(s_n)$.*

Definition 7.10. *Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a finite sample space. We say that it is a* uniform sample space *if $P(s_i) = 1/n$ for all $1 \leq i \leq n$.*

Example 7.11. An ice chest at a Texas A&M versus Alabama tailgate contains three brands of soda, each with a regular and diet option. A thirsty tailgater reaches into the chest and selects a drink according to the following table:

	Coke	Dr. Pepper	Pepsi
Regular	0.05	0.15	0.23
Diet	0.10	0.17	0.30

Find the probabilities of the following events:

1. P(regular Dr. Pepper),
2. P(Coke),
3. P(diet drink),
4. P(not Dr. Pepper),
5. $P(\{\text{regular Pepsi}\} \cup \{\text{diet Dr. Pepper}\})$.

Solution: We have:

1. $P(\text{regular Dr. Pepper}) = 0.15$,
2. $P(\text{Coke}) = 0.15$,
3. $P(\text{diet drink}) = 0.57$,
4. $P(\text{not Dr. Pepper}) = 0.68$,
5. $P(\{\text{regular Pepsi}\} \cup \{\text{diet Dr. Pepper}\}) = 0.40$.

Example 7.12. A pair of four sided die are rolled and the sum of the face up values is recorded.

1. Find the probability distribution for this experiment.
2. What is the probability that the sum is eight?
3. What is the probability that the sum is even?

Solution: We have:

1. Let x be the sum of the face up values.

x	2	3	4	5	6	7	8
$P(x)$	1/16	2/16	3/16	4/16	3/16	2/16	1/16

2. $P(8) = 1/16$.
3. $P(\{x\,\text{even}\}) = 8/16 = 1/2$.

Example 7.13. A pair of six sided die are rolled.

1. Find the probability that the two dice show the same value.
2. Find the probability that the sum of the numbers of the two dice is 6.

Solution: We have:

1. $P(\text{two dice show the same value}) = 6 \cdot 1/36 = 1/6$.

2. $P(\text{the sum of the numbers of the two dice is six}) = 5 \cdot 1/36 = 5/36$.

Example 7.14. A bag contains ten marbles: three red, four green, and three blue. Calculate the probability that the three marbles chosen will be:

1. All red, if marbles are drawn then replaced.

2. Two green and one blue, if marbles are drawn without replacement.

3. One of each color, if marbles are drawn then replaced.

Solution: We have:

1. $P(\text{all red with replacement}) = 3/10 \cdot 3/10 \cdot 3/10 = 27/1000$.

2. $P(\text{two gren and one blue without replacement}) = (4 \cdot 3 \cdot 3)/(10 \cdot 9 \cdot 8) + (4 \cdot 3 \cdot 3)/(10 \cdot 9 \cdot 8) + (3 \cdot 4 \cdot 3)/(10 \cdot 9 \cdot 8) = 3/20$.

3. $P(\text{one of each color with replacement}) = 6 \cdot 4/10 \cdot 3/10 \cdot 3/10 = 63/500$.

7.3 Rules of Probability

Proposition 7.15. *Let $\mathcal{S}$ be a sample space with E and F two events. Then:*

$$P(E \cup F) = P(E) + P(F) - P(E \cup F).$$

Proposition 7.16. *Let $\mathcal{S}$ be a sample space with E an events. Then:*

$$P(E^c) = 1 - P(E).$$

Example 7.17. A card is drawn from a standard deck of playing cards.

1. What is the probability that it is an ace or a spade?

2. What is the probability that it is a diamond or a face card?

3. What is the probability that it is a diamond or a club?

Solution: We have:

1. $P(\text{ace or space}) = P(\text{ace}) + P(\text{space}) - P(\text{ace of spades}) = 4/13$.

2. $P(\text{diamond or face}) = P(\text{diamond}) + P(\text{face}) - P(\text{faces of diamond}) = 11/26$.

3. $P(\text{diamond or club}) = P(\text{diamond}) + P(\text{club}) = 1/2$.

Example 7.18. The quality control department of Apple has determined that 3% of the iPhone sold experience video problems, 1% experience audio problems, and 0.1% experience both video and audio problems.

1. What is the probability that an iPhone experiences problem(s) related to video or audio?

2. What is the probability that an iPhone does not experience any problem(s) related to video or audio?

Solution: We have:

1. $P(\text{problem(s) related to video or audio}) = 3.9\%$.

2. $P(\text{no problem(s) related to video or audio}) = 96.1\%$.

Example 7.19. Let E and F be two mutually exclusive events with $P(E) = 0.1$ and $P(F) = 0.6$. Compute:

1. $P(E \cap F)$.

2. $P(E \cup F)$.

3. $P(E^c)$.

4. $P(E^c \cap F^c)$.

5. $P(E^c \cup F^c)$.

Solution: We have:

1. $P(E \cap F) = 0$.

2. $P(E \cup F) = 0.7$.

3. $P(E^c) = 0.9$.

4. $P(E^c \cap F^c) = 0.3$.

5. $P(E^c \cup F^c) = 1$.

Example 7.20. Let E and F be two events of an experiment with sample space $\mathcal{S}$. Suppose $P(E) = 0.2$, $P(F) = 0.1$, and $P(E \cap F) = 0.05$. Compute:

1. $P(E \cup F)$.

2. $P(E^c \cap F^c)$.

3. $P(E^c \cup F)$.

4. $P(E^c \cap F)$.

Solution: We have:

1. $P(E \cup F) = 0.25$.
2. $P(E^c \cap F^c) = 0.75$.
3. $P(E^c \cup F) = 0.85$.
4. $P(E^c \cap F) = 0.05$.

Example 7.21. A poll by the NRA regarding gun control laws was conducted among 250 Vermont residents that owned a handgun, a rifle, both, or neither. The results of the poll were:

	Only handgun	Only rifle	Both	Neither	Total
Favor tougher laws	0	12	0	138	150
Oppose tougher laws	58	5	25	0	88
No opinion	0	0	0	12	12
Total	58	17	25	150	250

If one of the participants in this poll is selected at random, what is the probability that they:

1. Favor tougher gun control laws.
2. Own a handgun.
3. Own a handgun but not a rifle.
4. Favor tougher gun control laws and does not own a handgun.

Solution: We have that the probability that a random participant:

1. Favor tougher gun control laws is: $150/250 = 3/5$.
2. Own a handgun is: $83/250$.
3. Own a handgun but not a rifle is: $58/250$.
4. Favor tougher gun control laws and does not own a handgun is: $138/250 = 69/125$.

7.4 Use of Counting Techniques in Probability

Theorem 7.22. *Given a finite uniform sample space $\mathcal{S} = \{s_1, \ldots, s_n\}$, an event E consisting of the sample points $e_1, \ldots, e_k$, and a probability $P : \mathcal{S} \longrightarrow [0, 1]$, then $P(E) = n(E)/n(S)$.*

Because of this result, it is enough to use the counting techniques for sets to compute the values for $n(E)$ and $n(S)$, and then dividing them we will obtain the probability of the event E happening.

Example 7.23. A fair coin is tossed six times. Compute the probability that the coin will land heads:

1. Exactly three times.
2. At most three times.
3. On the first and last toss.

Solution: We have that the probability that the coin will land heads:

1. Exactly three times is:
$$\frac{C(6,3)}{2^6} = \frac{20}{64} = \frac{5}{16}.$$
2. At most three times:
$$\frac{C(6,0)+C(6,1)+C(6,2)+C(6,3)}{2^6} = \frac{42}{64} = \frac{21}{32}.$$
3. On the first and last toss:
$$\frac{1 \cdot 2^4 \cdot 1}{2^6} = \frac{16}{64} = \frac{1}{4}.$$

Example 7.24. Two cards are selected at random from a standard deck of 52 cards. Compute the probability that:

1. They are both aces.
2. Exactly one is an ace.
3. Neither of them is an ace.

Solution: We have that the probability that:

1. They are both aces is:
$$\frac{C(4,2)}{C(52,2)} = \frac{6}{1326} = \frac{1}{221}.$$
2. Exactly one is an ace is:
$$\frac{C(4,1)C(48,1)}{C(52,2)} = \frac{192}{1326} = \frac{32}{221}.$$
3. Neither of them is an ace is:
$$\frac{C(48,2)}{C(52,2)} = \frac{1128}{1326} = \frac{188}{221}.$$

Example 7.25. At the MSC Lost and Found there are 100 blank cassette tapes, 10 of which are known to be defective. If a volunteer selects 6 of these cassette tapes, compute the probability that:

1. Two of them are defective.
2. At least one of them is defective.

Solution: We have that the probability that:

1. Two of them are defective is:
$$\frac{C(10,2)C(90,4)}{C(100,6)} = \frac{69687}{722456}.$$
2. At least one of them is defective is:
$$1 - \frac{C(90,6)}{C(100,6)} = \frac{1725569}{3612280}$$
since the probability that none of them is defective is $C(90,6)/C(100,6)$.

Example 7.26. A group of r people are selected at random from MATH 141:502. What is the probability that at least two of them have the same birthday?
Solution: We have that the probability that any two individuals have different birthdays is $C(365,r)/365^r$. This means that the probability that at least two of them have the same birthday is:
$$1 - \frac{C(365,r)}{365^r}.$$

Example 7.27. A five hand card of poker is dealt from a standard deck. Compute the probability of being dealt the following hands:

1. A straight flush.
2. Four of a kind.
3. A full house.
4. A flush (but not a straight flush).
5. A straight (but not a straight flush).
6. Three of a kind (but not a full house).
7. Two pair.

Solution: We have that the probability of being dealt:

1. A straight flush is:
$$\frac{C(10,1)C(4,1)}{C(52,2)} = \frac{40}{2598960}.$$
2. Four of a kind is:
$$\frac{C(13,1)C(4,4)C(48,1)}{C(52,2)} = \frac{624}{2598960}.$$

3. A full house is:
$$\frac{C(13,1)C(4,3)C(12,1)C(4,2)}{C(52,2)} = \frac{3744}{2598960}.$$

4. A flush (but not a straight flush) is:
$$\frac{C(4,1)C(13,5) - C(10,1)C(4,1)}{C(52,2)} = \frac{5108}{2598960}.$$

5. A straight (but not a straight flush) is:
$$\frac{C(10,1)C(4,1)C(4,1)C(4,1)C(4,1)C(4,1) - C(10,1)C(4,1)}{C(52,2)} = \frac{10200}{2598960}.$$

6. Three of a kind (but not a full house) is:
$$\frac{C(13,1)C(4,3)C(12,2)C(4,1)C(4,1)}{C(52,2)} = \frac{54912}{2598960}.$$

7. Two pair is:
$$\frac{C(13,2)C(4,2)C(4,2)C(44,1)}{C(52,5)} = \frac{123552}{2598960}.$$

7.5 Conditional Probability and Independent Events

Usually the probability of an event happening is affected by the occurrence of other events. We will use *conditional probability* to study this phenomenon.

Definition 7.28. *The probability that A occurs given that B has already occurred is called* conditional probability *and we denote it by $P(A|B)$.*

Example 7.29. Two cards are drawn without replacement from a well shuffled standard deck of 52 cards. Find:

1. The probability that the first card drawn is an ace.
2. The probability that the first card drawn is not an ace.
3. The probability that the second card drawn is an ace given that the first card drawn was an ace.
4. The probability that the second card drawn is an ace given that the first card drawn was not an ace.
5. The probability that the second card drawn is not an ace given that the first card drawn was an ace.

6. The probability that the second card drawn is not an ace given that the first card drawn was not an ace.

Solution: We may call A the event of the first card drawn being an ace and B the event of the second card drawn being an ace. We then have that:

1. $P(A) = 4/52$.

2. $P(A^c) = 48/52$.

3. $P(B|A) = 3/51$.

4. $P(B|A^c) = 4/51$.

5. $P(B^c|A) = 48/51$.

6. $P(B^c|A^c) = 47/51$.

Theorem 7.30. *Given A and B events in an experiment with $P(A) \neq 0$, then the probability that B will occur given that A has occurred is:*

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

The probability of event A and event B happening is then:

$$P(A \cap B) = P(B|A)P(A).$$

Example 7.31. A pair of fair 6 sided dice is rolled. What is the probability that the sum of the numbers facing up is 7 if it is known that one of the numbers is a 5?
Solution: We may call F the event of one of the numbers being a 5 and S the event that the sum of the numbers facing up is a 7. We then have that:

$$P(S|F) = \frac{P(S \cap F)}{P(F)} = \frac{2/36}{11/36} = \frac{2}{11}.$$

Example 7.32. In a test conducted by the US Army it was found that of 1000 new recruits, there were 600 with brown hair and 400 with blond hair. Moreover, 50 of the brown haired and 4 of the blond recruits were colorblind. Given that a randomly selected recruit is colorblind, what is the probability that the recruit has brown hair?
Solution: We may call C the event of a randomly selected recruit being colorblind and B the event that a randomly selected recruit has brown hair. We then have that:

$$P(B|C) = \frac{P(B \cap C)}{P(B)} = \frac{50/1000}{54/1000} = \frac{25}{27}.$$

Example 7.33. There are 300 seniors at Texas A&M Consolidated High School, of which 140 have brown hair, and the rest have blond hair. It is known that 80% of the brown haired seniors have a driver's license, while 60% of the blond haired seniors have a driver's license. Find:

1. The probability that the student has brown hair and a driver's license.
2. The probability that the student has blond hair but not a driver's license.
3. The probability that the student has a driver's license.

Solution: We may call B the event of a senior having brown hair and D the event of a senior having a driver's license. We then have that:

1. The probability that the student has brown hair and a driver's license is:
$$P(B \cap D) = \frac{140}{300} \cdot \frac{80}{100} = \frac{28}{75}.$$
2. The probability that the student has blond hair but not a driver's license is:
$$P(B^c \cap D^c) = \frac{160}{300} \cdot \frac{40}{100} = \frac{16}{75}.$$
3. The probability that the student has a driver's license is:
$$P(D) = P(D \cap B) + P(D \cap B^c) = \frac{140}{300} \cdot \frac{80}{100} + \frac{160}{300} \cdot \frac{60}{100} = \frac{52}{75}.$$

Example 7.34. Two cards are drawn without replacement from a well shuffled standard deck of 52 cards. Find:

1. The probability that the first card drawn is an ace and the second card drawn is an ace.
2. The probability that the first card drawn is not an ace and the second card drawn is an ace.

Solution: We may call A the event of the first card drawn being an ace and B the event of the second card drawn being an ace. We then have that:

1. $P(A \cap B) = (4/52) \cdot (3/51) = 1/221$.
2. $P(A^c \cap B) = (48/52) \cdot (4/51) = 16/221$.

Example 7.35. A box contains eight batteries, two of which are known to be defective. The batteries are selected (one at a time without replacement) and tested until a non-defective one is found. What is the probability that the number of batteries tested is:

1. One.
2. Two.
3. Three

Solution: We have that:

1. $P(\{\text{one battery tested}\}) = 6/8 = 3/4$.
2. $P(\{\text{two battery tested}\}) = (2/8) \cdot (6/7) = 3/14$.
3. $P(\{\text{two battery tested}\}) = (2/8) \cdot (1/7) \cdot 1 = 1/28$.

Definition 7.36. *We say that two events A and B are* independent *if the outcome of one event does not affect the outcome of the other, that is, $P(B|A) = P(B)$ and $P(A|B) = P(A)$. The probability of the independent events A and B happening is then:*

$$P(A \cap B) = P(B)P(A)$$

and two events A and B are independent if and only if $P(A \cap B) = P(B)P(A)$.

Example 7.37. An experiment consists of two trials. The outcomes of the first trial are X, Y, and Z with probabilities 0.3, 0.25, and 0.45 respectively. The outcomes of the second trial are A and B with probabilities 0.3 and 0.7 respectively.

1. Find $P(X) \cdot P(B)$.
2. Find $P(X \cap B)$.
3. Are X and B mutually exclusive?
4. Are X and B independent?

Solution: We have that:

1. $$\begin{aligned} P(X) &= 0.3 \\ P(B) &= P(B \cap X) + P(B \cap Y) + P(B \cap Z) \\ &= 0.3 \cdot 0.7 + 0.25 \cdot 0.7 + 0.45 \cdot 0.7 = 0.7 \\ P(X) \cdot P(B) &= 0.3 \cdot 0.7 = 0.21. \end{aligned}$$
2. $P(X \cap Y) = (0.3) \cdot (0.7) = 0.21$.
3. No since $X \cap B \neq \emptyset$.
4. Yes since $P(B \cap X) = P(X) \cdot P(Y)$.

Example 7.38. A survey conducted for the National Science Foundation found that of 2000 individuals, 680 were heavy smokers, 50 had emphysema, and 42 of the heavy smokers had emphysema. Using this data, determine whether the events S "being a heavy smoker" and E "having emphysema" are independent events.
Solution: We need to determine whether $P(S \cap E)$ equals $P(S) \cdot P(E)$. We have:

$$P(S \cap E) = 42/2000$$

and

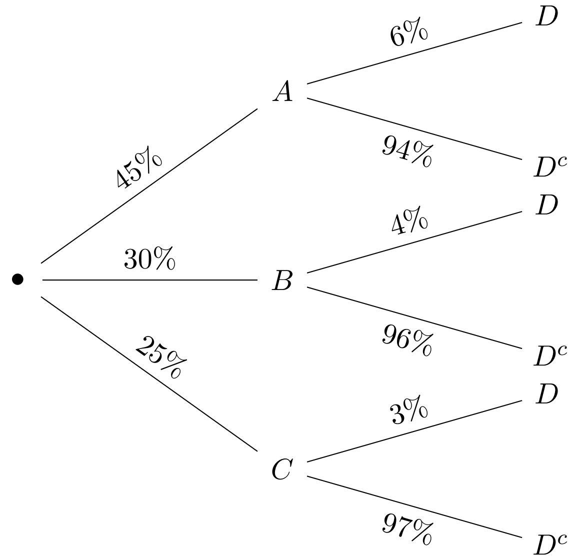
$$P(S) = 680/2000, \quad P(E) = 50/2000 \quad \text{so} \quad P(S) \cdot P(E) = 17/2000$$

and thus since $P(S \cap E) 42/2000 \neq 17/2000 = P(S) \cdot P(E)$ we have that they are not independent events.

7.6 Bayes' Theorem

Sometimes we want to compute probabilities after the outcome of an experiment has been observed. These are called *a posteriori probabilities*. The probabilities that give the likelihood that an event will occur in the future are called *a priori probabilities*.

Example 7.39. Three machines A, B, and C produce the same engine component. Machine A produces 45% of the total components, machine B produces 30% of the total components, and machine C produces 25% of the total components. However, 6% of the components produced by machine A do not meet established specifications, 4% of the components produced by machine B do not meet established specifications, and 3% of the components produced by machine A do not meet established specifications. A component was randomly selected and was found to be defective, call this event D. What is the probability that the component selected was produced by machine A?
Solution: Consider the following tree diagram:



We can read from it that:

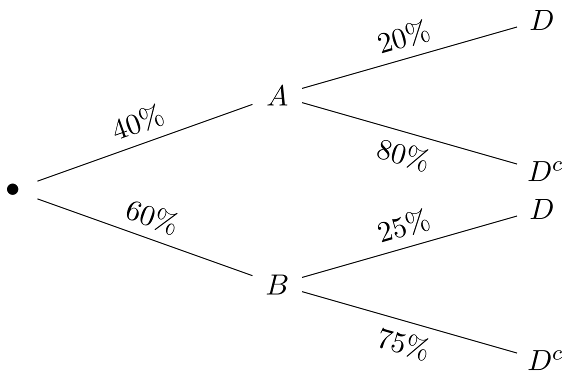
$$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{0.45 \cdot 0.06}{0.45 \cdot 0.06 + 0.3 \cdot 0.04 + 0.25 \cdot 0.03} = \frac{18}{31}.$$

Theorem 7.40. *Let A_1, ...,A_n be a partition of the sample space $\mathcal{S}$ and let E be an event of the experiment such that $P(E) \neq 0$. Then the a posteriori probability $P(A_i|E)$, $1 \leq i \leq n$, is given by:*

$$P(A_i|E) = \frac{P(A_i)P(E|A_i)}{P(A_1)P(E|A_1) + \cdots + P(A_n)P(E|A_n)} = \frac{P(A_i)P(E|A_i)}{\sum_{j=1}^{n} P(A_j)P(E|A_j)}.$$

Example 7.41. Use the following tree diagram to find:

1. $P(A)P(D|A)$.
2. $P(B)P(D|B)$.
3. $P(A|D)$.



Solution: We have:

1. $$P(A)P(D|A) = P(A)\frac{P(A \cap D)}{P(A)} = 0.40 \cdot 0.20 = \frac{2}{25}.$$

2. $$P(B)P(D|B) = P(B)\frac{P(B \cap D)}{P(B)} = 0.60 \cdot 0.25 = \frac{3}{20}.$$

3. $$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{0.40 \cdot 0.20}{0.40 \cdot 0.20 + 0.60 \cdot 0.25} = \frac{8}{23}.$$

Example 7.42. The trademark Texas A&M University merchandise is manufactured in three different locations and then shipped to College Station for retail. Bryan, Navasota, and Waco provide 50%, 30%, and 20% respectively of the merchandise. The quality control over time has determined that 1% of the products from Bryan are defective, while 2% of the products from Navasota and Waco are defective. If a random item is selected and it is found to be defective, what is the probability that it was manufactured in Waco?
Solution: We have that:

$$P(W|D) = \frac{P(W \cap D)}{P(D)} = \frac{0.20 \cdot 0.02}{0.50 \cdot 0.01 + 0.30 \cdot 0.02 + 0.20 \cdot 0.02} = \frac{4}{15}.$$

Example 7.43. The office of the Registrar released the accompanying information concerning the contemplated majors of its freshman class, and if they had prior experience in the major or not. The following table summarizes that information:

Major	% freshman choosing	% experience	% no experience
Business	24	38	62
Humanities	8	60	40
Education	8	66	34
Social Science	7	58	42
Natural Sciences	9	52	48
Other	44	48	52

1. What is the probability that a student selected at random from the freshman class had prior experience in their major?

2. What is the probability that a business student selected at random from the freshman class had no prior experience in their major?

3. What is the probability that a student with prior experience in their major selected at random from the freshman class is majoring in business?

Solution: We call X the event of a student having prior experience in their major, and Y the event of a student not having prior experience in their major. We then have:

1.

$$\begin{aligned} P(X) &= P(X \cap B) + P(X \cap H) + P(X \cap E) + P(X \cap S) + P(X \cap N) + P(X \cap O) \\ &= 0.24 \cdot 0.38 + 0.08 \cdot 0.60 + 0.08 \cdot 0.66 + 0.07 \cdot 0.58 + 0.09 \cdot 0.52 + 0.44 \cdot 0.48 \\ &= 0.4906. \end{aligned}$$

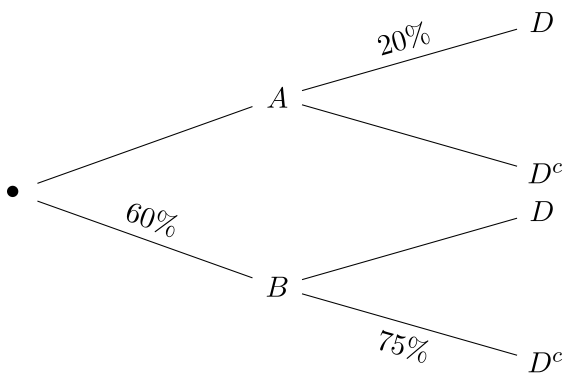
2.

$$P(Y|B) = \frac{P(Y \cap B)}{P(B)} = \frac{0.24 \cdot 0.62}{0.24} = \frac{31}{50}.$$

3.
$$P(B|X) = \frac{P(B \cap X)}{P(X)} = \frac{0.24 \cdot 0.38}{0.4906} = \frac{456}{2453}.$$

Example 7.44. Find the missing probabilities on the following tree diagram and then find:

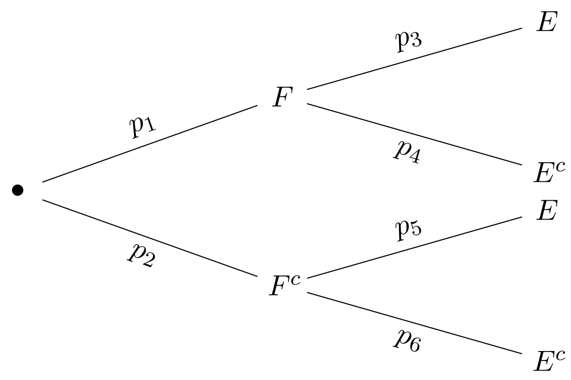
1. $P(A)$.
2. $P(D)$.
3. $P(D^c)$.
4. $P(B \cap D)$.
5. $P(D|A)$.
6. $P(B|D^c)$.



Solution: We have:

1. $P(A) = 0.4$.
2. $P(D) = 0.23$.
3. $P(D^c) = 0.77$.
4. $P(B \cap D) = 0.15$.
5. $P(D|A) = 0.20$.
6. $P(B|D^c) = 45/77$.

Example 7.45. Given $P(F)$ and $P(E|F)$, find $P(F|E)$ in the context of the following tree diagram.



Solution: We have:

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{p_1 \cdot p_3}{p_1 \cdot p_3 + p_2 \cdot p_5} = \frac{P(F)P(E|F)}{P(F)P(E|F) + P(F^c)P(E|F^c)}$$
$$= \frac{P(F)P(E|F)}{P(F)P(E|F) + (1 - P(F))P(E|F^c)}$$

so we also need information about $P(E|F^c)$ to be able to find what we are asked.

Example 7.46. We have two jars of cookies. In one we have 10 chocolate chip cookies and 30 plain cookies, while in the second we have 20 of each. We toss a coin to determine from which jar we are going to pick, and then we randomly choose a cookie from that jar. In we chose a plain cookie, what is the probability that we picked the cookie from the first jar?
Solution: We have that:

$$P(J1|P) = \frac{P(J1 \cap P)}{P(P)} = \frac{0.50 \cdot (30/40)}{0.50 \cdot (30/40) + 0.50 \cdot (20/40)} = \frac{3}{5}.$$

8 Probability Distributions and Statistics

Statistics is the branch of mathematics concerned with the collection, analysis, and interpretation, of data.

8.1 Distributions of Random Variables

Definition 8.1. *A* random variable *is a function $X : \mathcal{S} \longrightarrow \mathbb{R}$ that assigns a real number to each outcome of a chance experiment.*

Example 8.2. A coin is tossed three times. Let the random variable X denote the number of heads that occur in three tosses.

1. List the outcomes of the experiment.
2. Find the value assigned to each outcome of the experiment by the random variable X.
3. Find the event comprising the outcomes to which a value of two has been assigned by X, that is, $X = 2$.

Solution:

1. The sample space is:
$$\mathcal{S} = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$
2. The values assigned by X are the following table:

$\mathcal{S}$	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
X	3	2	2	2	1	1	1	0

3. The event is $\{HHT, HTH, THH\}$.

Example 8.3. A coin is tossed repeatedly until a head occurs. Let Y be the random variable denoting the number of coin tosses in the experiment. What are the values of Y?
Solution: $Y \in \mathbb{N}$, $Y \geq 1$.

Example 8.4. A disposable flashlight is turned on until the battery runs out. Let the random variable Z denote the length in hours of the life of the battery. What are the values that Z may assume?
Solution: $Z \in \mathbb{R}$, $Z \geq 0$.

Example 8.5. A car has a full tank. It is driven until the tank is empty. Let the random variable Z denote the distance in kilometers that the car traveled. What are the values that Z may assume?
Solution: $Z \in \mathbb{R}$, $Z > 0$.

Definition 8.6. *Given a random variable $X : \mathcal{S} \longrightarrow \mathbb{R}$, it is classified into three categories depending on the values it assumes.*

1. Finite discrete*: assumes only finitely many values.*
2. Infinite discrete*: assumes infinitely many values that can be arranged in a sequence.*
3. Continuous*: assumes values that comprise an interval of the real numbers.*

Definition 8.7. *Given a random variable $X : \mathcal{S} \longrightarrow \mathbb{R}$, its* probability distribution *is a table displaying the outcomes of the random variable together with the probability of that outcome happening.*

Example 8.8. A coin is tossed three times. Let the random variable X denote the number of heads that occur in three tosses.

1. Find the probability distribution of the random variable.
2. Find the probability of $X \geq 2$.

Solution:

1. The probability distribution is:

X	0	1	2	3
$P(X)$	1/8	3/8	3/8	1/8

2. We have: $P(X \geq 2) = P(X = 2) + P(X = 3) = 1/2$.

Example 8.9. Once a minute for an hour, an observer counts the number of cars waiting in line at McDonald's. The following data give the number of cars observed and the frequency of occurrence. Find the probability distribution of the random variable X, where X is the number of cars waiting in line.

Number of cars	0	1	2	3	4	5	6	7	8
Frequency of occurrence	2	9	16	12	8	6	4	2	1

Solution: The probability distribution is:

X	0	1	2	3	4	5	6	7	8
$P(X)$	2/60	9/60	16/60	12/60	8/60	6/60	4/60	2/60	1/60

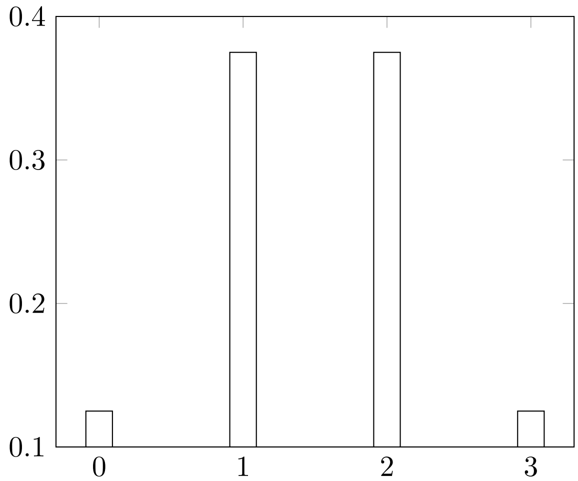
Definition 8.10. *A* histogram *is a graphical representation of the distribution of a random variable. The following is a method to construct a histogram:*

1. *Draw a number line that contains all the variables attained by the random variable.*

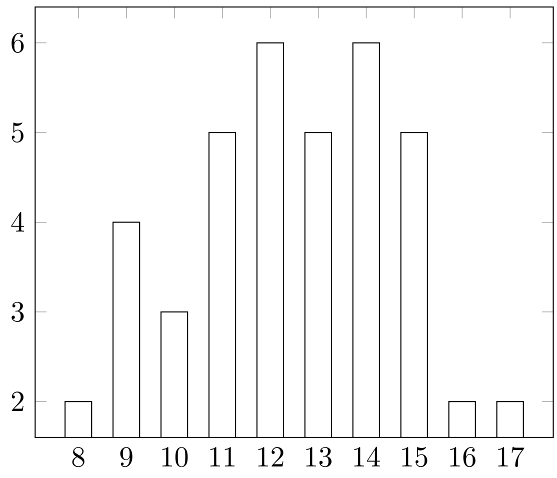
2. *Above each value attained by the random variable, draw a rectangle of width one and height equal to the probability that the random variable assigns to that value.*

In this way, the area of the rectangle associated with a value attained by the random variable gives exactly the probability that the random variable associates to that value. In addition, the probability that the random variable associates to several value is given by the sum of the areas of the rectangles associated with those values that the random variable takes.

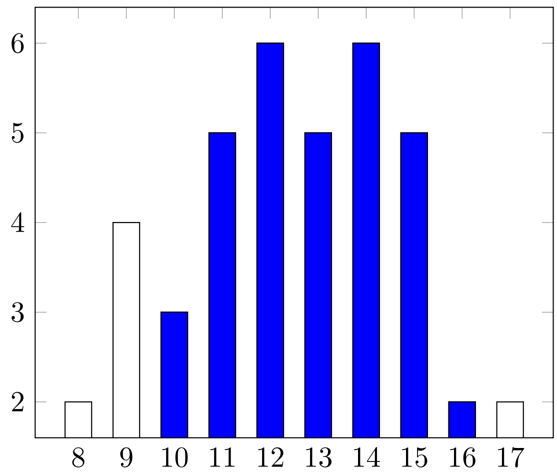
Example 8.11. A coin is tossed three times. Let the random variable X denote the number of heads that occur in three tosses. Construct a histogram for the random variable X.
Solution: We have:



Example 8.12. Suppose the probability distribution of a random variable X is represented by the following non-normalized histogram. Shade the part of the histogram whose area gives the probability $P(10 \leq X \leq 16)$.



Solution: We shade:



8.2 Expected Value

Definition 8.13. *The* average *or* mean *of the* n *numbers* $x_1, \ldots, x_n$ *is:*

$$\overline{x} = \frac{x_1 + \cdots + x_n}{n}.$$

The median *of a collection of numbers arranged in increasing or decreasing order is the middle number if there is an odd number of entries, and the mean of the two middle numbers if there is an even number of entries.*

The mode *of a collection of numbers is the number that occurs more frequently in the collection.*

These three numbers are sometimes referred as *measures of central tendencies.*

Example 8.14. Once a minute for an hour, an observer counts the number of cars waiting in line at McDonald's. The following data give the number of cars observed and the frequency of occurrence. Find the three measures of central tendencies.

Number of cars	0	1	2	3	4	5	6	7	8
Frequency of occurrence	2	9	16	12	8	6	4	2	1

Solution: We have $\overline{x} = 185/60 = 37/12$, the median is 3 (the average of positions 30 and 31, which are both 3), and the mode is 2.

Example 8.15. Once a minute for an hour, an observer counts the number of cars waiting in line at McDonald's. The following data give the number of cars observed and the frequency of occurrence. Find the probability distribution of the random variable X, where X is the number of cars waiting in line. Find the average number of cars waiting in line at McDonald's.

Number of cars	0	1	2	3	4	5	6	7	8
Frequency of occurrence	2	9	16	12	8	6	4	2	1

Solution: The probability distribution is:

X	0	1	2	3	4	5	6	7	8
$P(X)$	2/60	9/60	16/60	12/60	8/60	6/60	4/60	2/60	1/60

The average number of cars is $\overline{x} = 185/60 = 37/12$.

Definition 8.16. *Let X be a random variable assuming the values $x_1, \dots, x_n$ with associated probabilities $p_1, \dots, p_n$ respectively. The* expected value *of X is denoted by $E(X)$ and is given by:*

$$E(X) = x_1p_1 + \cdots + x_np_n.$$

Example 8.17. Two dice are rolled. Let X be a random variable giving the sum of the faces that fall uppermost. Find the expected value of X.
Solution: The probability distribution is:

X	2	3	4	5	6	7	8	9	10	11	12
$P(X)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

The expected value is $E(X) = 7$.

Example 8.18. The Island Club is holding a fund raising raffle. They have sold 10000 for \$2 each. There will be a first price of \$3000, three second prices of \$1000 each, five third prices of \$500 each, and twenty consolation prices of \$100 each. Let X denote the net winnings of a ticket. Find the probability distribution of X and the expected value of X. Interpret your results.
Solution: The probability distribution is:

X	-2	98	498	998	2998
$P(X)$	9971/10000	20/10000	5/10000	3/10000	1/10000

The expected value is $E(X) = -0.95$. That is, we expect to lose \$0.95 per ticket bought.

Example 8.19. A group of Wall Street investors intend to purchase one of two motel complexes for sale in New York City. The terms of sale of the two motels are similar, but the Days Inn has 52 rooms and is in a slightly better location than the Holiday Inn, which has 60 rooms. The records from each motel for the occupancy rates and the corresponding probabilities between May and September are shown in the tables below. The average profit per day for each room at the Days Inn is \$10, while the average profit per day for each room at the Holiday Inn is \$9.

Days Inn					
Occupancy	80%	85%	90%	95%	100%
Probability	0.19	0.22	0.31	0.23	0.05

Holiday Inn						
Occupancy	75%	80%	85%	90%	95%	100%
Probability	0.35	0.21	0.18	0.15	0.09	0.02

1. Find the average number of rooms occupied per day at each motel.
2. If the investor's objective is to purchase the motel that generates the higher daily profit, which motel should they purchase?

Solution:

1. The expected occupancy rate for each motel is:

$$E(D) = 0.8865 \quad \text{and} \quad E(H) = 0.824$$

so the average number of rooms occupied is:

$$D\ :\ 46.098 \approx 46 \quad \text{and} \quad H\ :\ 49.44 \approx 49.$$

2. The daily average profit is then:

$$D\ :\ 460.98 \approx 461 \quad \text{and} \quad H\ :\ 444.96 \approx 445.$$

so the investors should purchase the Daily Inn.

Definition 8.20. *Let $P(E)$ be the probability of an event E happening. The* odds in favor *of E happening are $P(E)/P(E^c) = P(E)/(1 - P(E))$. The* odds against *$E$ happening are $P(E^c)/P(E) = (1 - P(E))/P(E)$.*

Example 8.21. The probability of an event E of not occurring is 0.6. What are the odds in favor of E occurring? What are the odds against E occurring?
Solution: The odds in favor of E occurring are 2/3. The odds against E occurring are 3/2.

Definition 8.22. *Let the odds in favor of an event E occurring be a to b. Then the probability of the event E occurring is $P(E) = a/(a+b)$. This is described as* given the odds.

Example 8.23. The odds that the Dallas Cowboys win the Superbowl this season are 7 to 5. What is the probability of the event occurring?
Solution: The probability of the event occurring is 7/12.

Example 8.24. The odds that it will not rain tomorrow are 3 to 2. What is the probability of rain tomorrow?
Solution: The probability of rain tomorrow is 2/5.

Example 8.25. The probability of rain tomorrow is 20%. What are the odds that it will not rain tomorrow?
Solution: We denote by E the event of raining tomorrow, so E^c is the event of not raining tomorrow. We are given that $P(E) = 0.2$. This means that $P(E^c) = 1 - 0.2 = 0.8 = 4/5$. Since if the odds in favor of E^c occurring are a to b, then the probability of E^c occurring is $a/(a+b) = P(E^c) = 4/5$, we have that $a = 4$ and $b = 1$. This means that the odds in favor of E^c occurring are 4 to 1.

8.3 Variance and Standard Deviation

The mean, as we saw, expresses the location of the center of a probability distribution. The *variance* of a probability distribution measures the spread of the values that the probability distribution takes around the mean.

Definition 8.26. *Suppose that a random variable has the following probability distribution.*

x	x_1	$\cdots$	x_n
$P(X = x)$	p_1	$\cdots$	p_n

with expected value $E(X) = \mu$. Then the variance *of X is:*

$$\mathrm{Var}(X) = p_1(x_1 - \mu)^2 + \cdots + p_n(x_n - \mu)^2.$$

The standard deviation *of X is $\sigma = \sqrt{\mathrm{Var}(X)}$.*

Example 8.27. A random variable X has the following probability distribution. Find the mean, the variance, and the standard deviation.

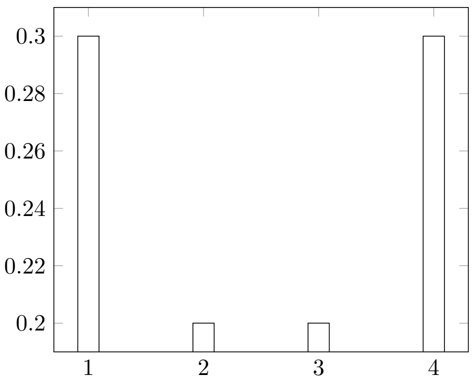
X	1	2	3	4	5
$P(X)$	0.3	0.2	0.1	0.2	0.2

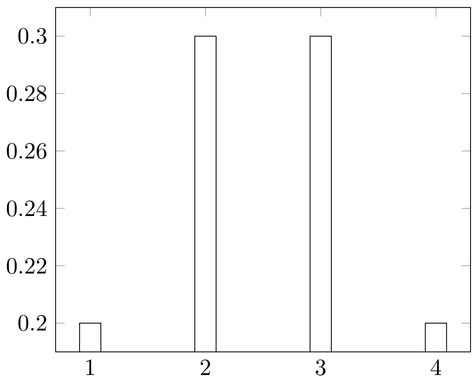
Solution: The expected value is $E(X) = 2.8$. To find the variance we complete the following table.

x_i	$x_i - \mu$	$(x_i - \mu)^2$	$p_i(x_i - \mu)^2$
1	-1.8	3.24	0.972
2	-0.8	0.64	0.128
3	0.2	0.04	0.004
4	1.2	1.44	0.288
5	2.2	4.84	0.968
Total			2.36

and thus $\text{Var}(X) = 2.36$ meaning that $\sigma = \sqrt{2.36} \approx 1.536$.

Example 8.28. The following histograms represent the probability distribution of the random variables X and Y respectively.





1. Determine the mean of each random variable. Determine by inspection which probability distribution has the larger variance.

2. Find the variance and standard deviation for each random variable.

Solution:

1. We have $E(X) = 2.5 = E(Y)$. Since the values of X are more disperse, and the values of Y are more concentrated, the variance of X should be larger than the variance of Y.

2. We have $\text{Var}(X) = 1.45$ and $\sigma_X = \sqrt{145}/10 \approx 1.204$, $\text{Var}(Y) = 1.05$ and $\sigma_Y = \sqrt{105}/10 \approx 1.025$.

8.4 The Binomial Distribution

An important type of experiments are the ones that have exactly two outcomes: a coin is tossed and can land in heads or tails, a game of chance is played and the player can win or lose, a medical treatment can be effective or ineffective... This type of experiments are called *Bernoulli trials* or *binomial trials*.

Definition 8.29. *A* binomial experiment *is one having the following properties:*

1. *The number of trials in the experiment is fixed.*
2. *There are exactly two outcomes to the experiment: success and failure.*
3. *The probability of success in each trial is the same.*
4. *The trials are independent of each other.*

A random variable X that gives the number of successes in a binomial experiment is called a binomial random variable *and its distribution is called a* binomial distribution.

Example 8.30. Determine whether the following experiments are binomial. Justify your answer

1. Rolling a fair six sided dice three times and observing the number of times a 6 is obtained.
2. Rolling a fair six sided dice three times and observing the number that lands uppermost.
3. Recording the number of accidents that occur at a given intersection on four clear days and one rainy day.

Solution:

1. It is binomial since:
   (a) The number of trials in the experiment is fixed.
   (b) There are exactly two outcomes: rolling a 6 is a success and rolling everything else is a failure.
   (c) The probability of success in each trial is 1/6.
   (d) The trials are independent of each other.
2. It is not binomial since we have more than one outcome.
3. It is not binomial for several reasons: the number of accidents is not fixed (that is, we do not have a fixed number of trials), we have more than one outcome (that is, we do not have a success), and the probability of having an accident on a rainy day is not the same as the probability of having an accident on a clear day.

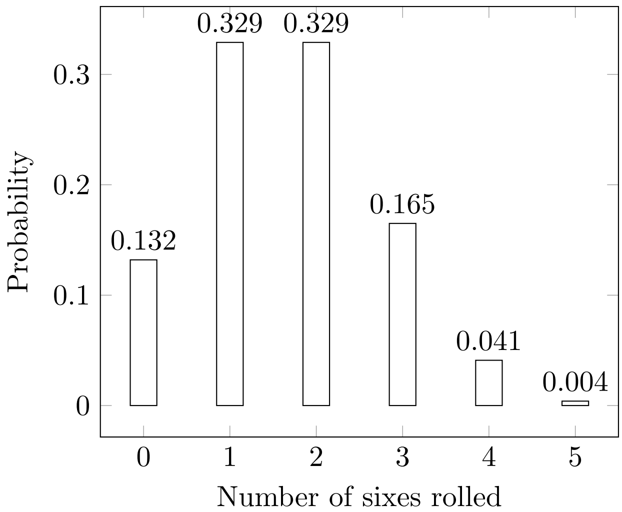
Theorem 8.31. *In a binomial experiment with probability of success p, set $q = 1-p$, then the probability of exactly x successes in n independent trials is $C(n,x)p^x q^{n-x}$.*

Example 8.32. A fair dice is rolled four times. Compute the probability of obtaining exactly one 6 in the four throws.
Solution: We call S the event of rolling a six, and F the event of not rolling a six. We can have:

$$\begin{array}{lll} SFFF & \text{with probability} & \frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6} \\ FSFF & \text{with probability} & \frac{5}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6} \\ FFSF & \text{with probability} & \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}\cdot\frac{5}{6} \\ FFFS & \text{with probability} & \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} \end{array}$$

and thus the probability is $4\cdot(1/6)\cdot(5/6) = C(4,1)(1/6)^1(5/6)^3 \approx 0.3858$.

Example 8.33. A player rolls a fair dice five times. If a one or a six is rolled, the player wins. Otherwise, the player loses. This experiment has the following histogram (approximated to the third decimal).



1. What is the probability of obtaining zero or one successes in the experiment?
2. What is the probability of obtaining at least one success in the experiment?

Solution:

1. The probability of obtaining zero or one successes in the experiment is $0.132 + 0.329 = 0.461$.

2. The probability of obtaining at least one success in the experiment is $1 - 0.132 = 0.868$.

Example 8.34. It was found that 30% of the restaurants in College Station are in violation of the health code. If a health inspector randomly selects five restaurants for inspection, what is the probability that:

1. None of the restaurants are in violation of the health code?
2. Just one of the restaurants is in violation of the health code?
3. At least two of the restaurants are in violation of the health code?

Solution: We have $p = 0.3$, so $q = 0.7$, and $n = 5$.

1. $P(X = 0) = C(5, 0)(0.3)^0(0.7)^5 = 0.16807$.
2. $P(X = 1) = C(5, 1)(0.3)^1(0.7)^4 = 0.36015$.
3. $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) = 0.47178$.

Example 8.35. A new drug being tested causes a serious side effect in 5 out of 100 patients. What is the probability that in a sample of 10 patients, none get the side effects from taking the drug?
Solution: The success of the experiment is getting the side effect. In that case we have $p = 0.05$, so $q = 0.95$, and $n = 10$. The probability of none getting the side effects is then $P(X = 0) = C(10, 0)(0.05)^0(0.95)^{10} \approx 0.5987$.

Theorem 8.36. *Let X be a binomial random variable associated with a binomial experiment consisting of n trials with probability of success p, and set $q = 1 - p$. Then: $E(X) = np$, $\mathrm{Var}(X) = npq$, and $\sigma_X = \sqrt{npq}$.*

Example 8.37. A player rolls a fair dice five times. If a one or a six is rolled, the player wins. Otherwise, the player loses. Find the mean, the variance, and the standard deviation of the binomial random variable associated to this experiment.
Solution: We have $p = 1/3$, so $q = 2/3$, and $n = 5$. Thus $E(X) = 5/3$, $\mathrm{Var}(X) = 5 \cdot (1/3) \cdot (2/3) = 10/9$, $\sigma_X = \sqrt{10}/3 \approx 1.0541$.

8.5 The Normal Distribution

A continuous random variable may have any value lying in an interval of real numbers, and thus their corresponding probability distributions are curves, that is, functions.

Definition 8.38. *The function associated with the probability distribution of a continuous random variable X is called a* probability density function.

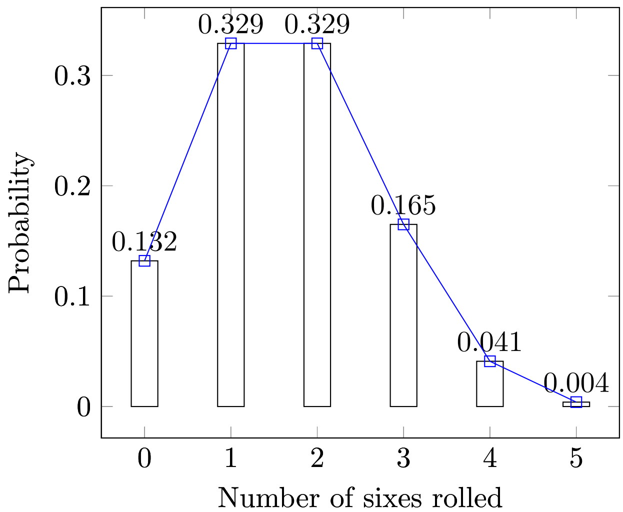
Theorem 8.39. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a probability density function associated with the random variable X. We then have:*

1. *The values $f(x)$ are non-negative for all $x \in \mathbb{R}$.*

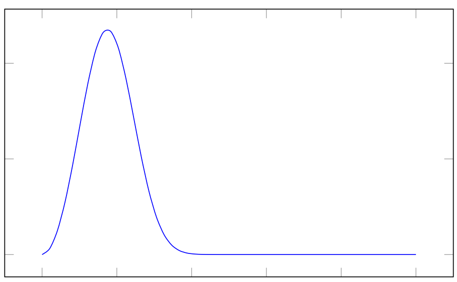
2. *The area of the region between the graph of f and the horizontal axis is 1.*

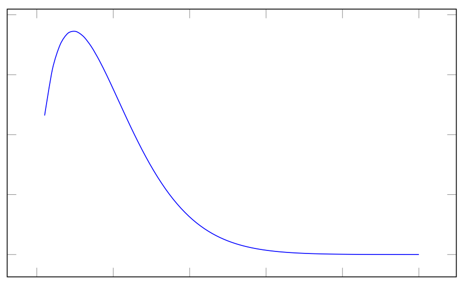
3. *The probability $P(a \leq X \leq b)$ is equal to the area of the region between the graph of f, the horizontal axis, the vertical line $x = a$, and the vertical line $x = b$.*

Recalling the experiment having a player rolling a fair dice five times, and whenever a one or a six is rolled the player wins, but otherwise the player loses. This can be represented in the following histogram, where the segments connecting the top of the bars have been connected.

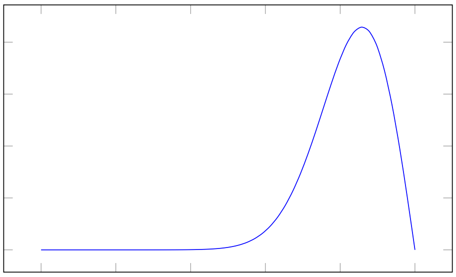
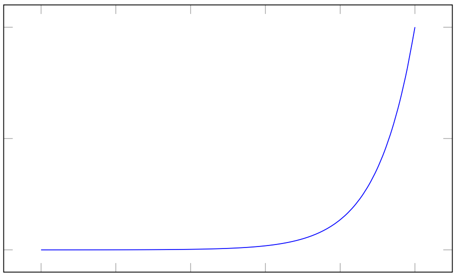


The following are left skewed continuous probability density functions:

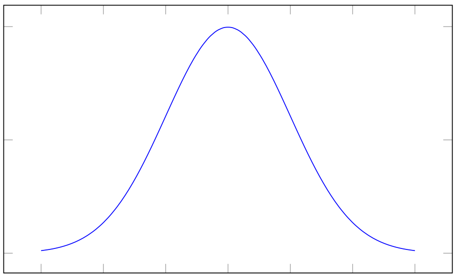
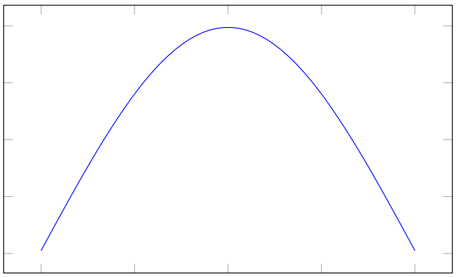




The following are right skewed continuous probability density functions:

The following are centered, or normal, continuous probability density functions:

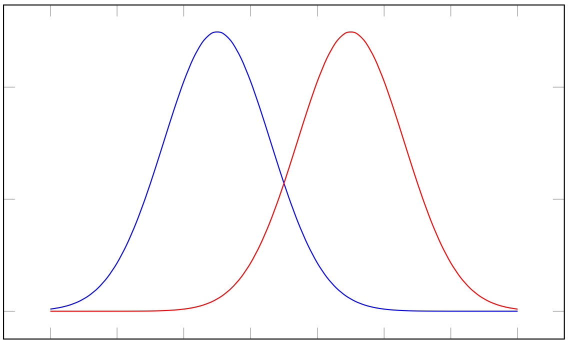
 

We will now focus on a specific family of continuous probability distributions, the *normal distributions*. These depend only on the parameters μ and σ, which are respectively their mean and standard deviation.

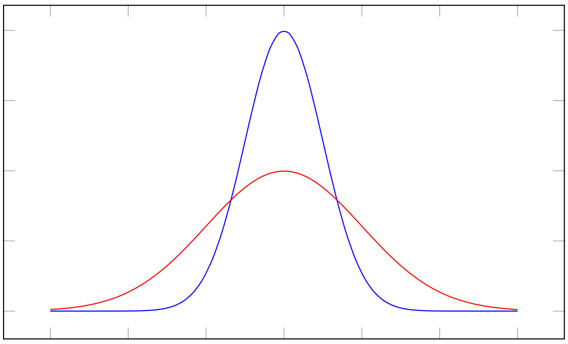
Theorem 8.40. *The following are true for any normal distribution:*

1. *The curve has a peak at* $x = \mu$.
2. *The curve is symmetric with respect to the vertical line* $x = \mu$.
3. *The curve always lies above the horizontal axis, and it approaches this axis as* x *extends indefinitely in either direction.*
4. *The area under the curvee is* 1.
5. *A* 68.72% *of the area under the curve lies between* $\mu - \sigma$ *and* $\mu + \sigma$. *A* 95.45% *of the area under the curve lies between* $\mu - 2\sigma$ *and* $\mu + 2\sigma$. *A* 99.73% *of the area under the curve lies between* $\mu - 3\sigma$ *and* $\mu + 3\sigma$.

The following normal distributions have different μ but the same σ.



The following normal distributions have the same μ but different σ.



Any normal distribution can be transformed into any other normal distribution, so we will only care about some specific values of μ and σ.

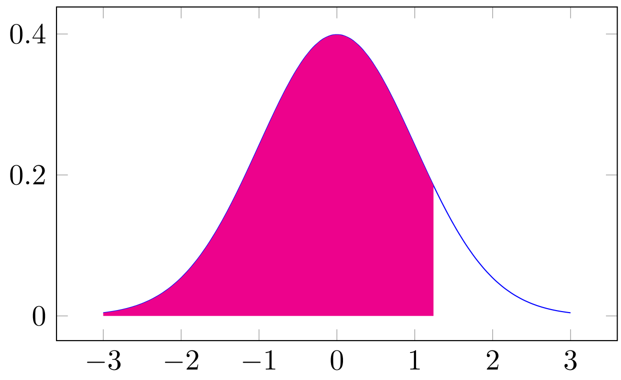
Definition 8.41. *The normal distribution defined by the values $\mu = 0$ and $\sigma = 1$ is called the* standard normal distribution. *The curve itself is called the* standard normal curve. *The random variable Z associated to this distribution is called the* standard normal variable.

Example 8.42. Let Z be the standard normal variable. Make a sketch of the appropriate region under the standard normal curve and find the values of:

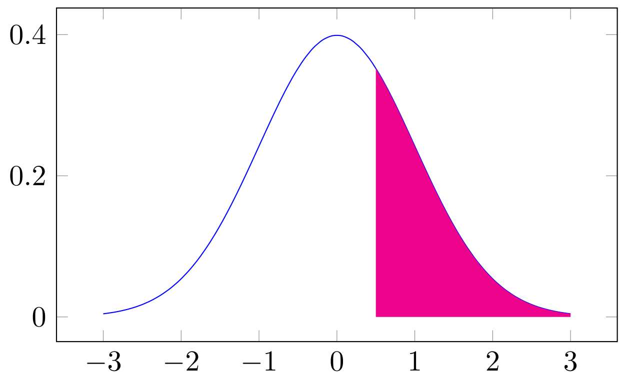
1. $P(Z < 1.24)$.
2. $P(Z > 0.5)$.
3. $P(0.24 < Z < 1.48)$.
4. $P(-1.65 < Z < 2.02)$.

Solution: We have:

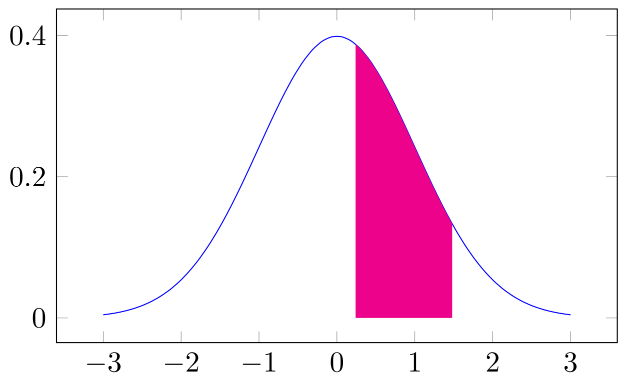
1. $P(Z < 1.24) \approx 0.8925$ and:



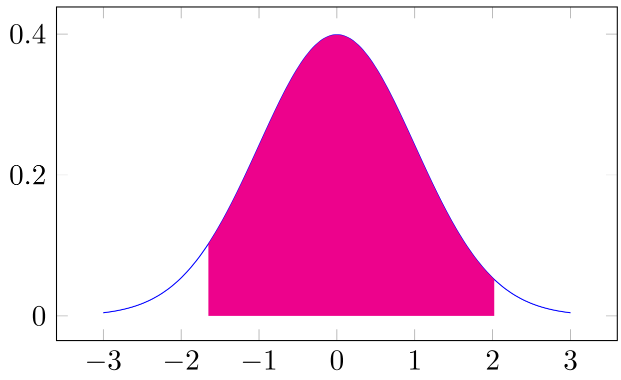
2. $P(Z > 0.5) \approx 0.3085$ and:



3. $P(0.24 < Z < 1.48) \approx 0.3358$ and:



4. $P(-1.65 < Z < 2.02) \approx 0.9288$ and:

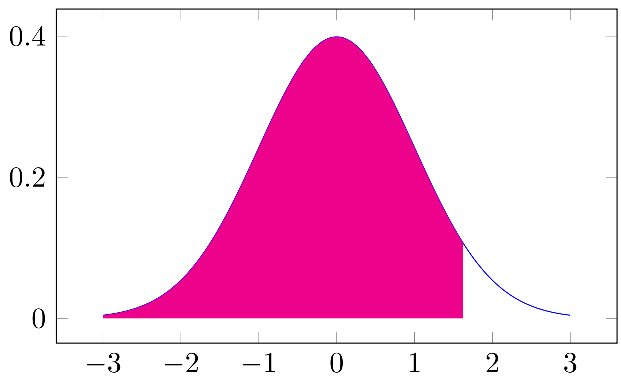


Example 8.43. Let Z be the standard normal variable. Find the value z satisfying:

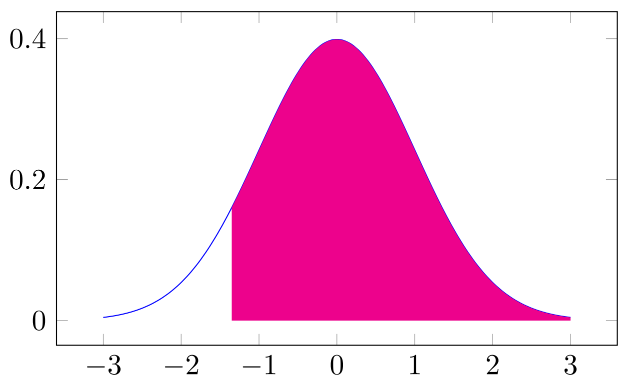
1. $P(Z < z) = 0.9474$.
2. $P(Z > z) = 0.9115$.
3. $P(-z < Z < z) = 0.7888$.

Solution: We have:

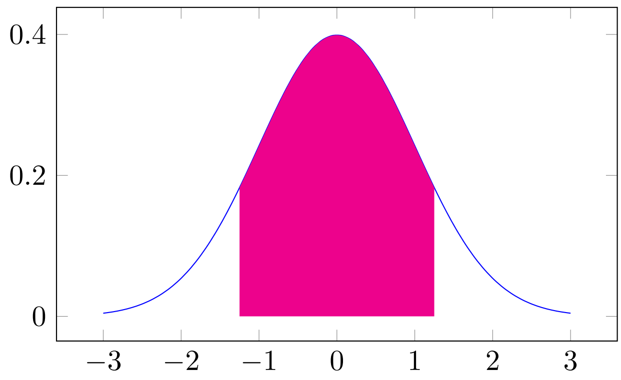
1. $z = 1.62$ and the graphic representation is:



2. $z = -1.35$ and the graphic representation is:



3. $z = 1.25$ and the graphic representation is:



We will now make precise the previous statement that "any normal distribution can be transformed into any other normal distribution".

Theorem 8.44. *Let X be a normal random variable with mean μ and standard deviation σ. Then it can be transformed into the standard normal random variable Z by the formula:*

$$Z = \frac{X - \mu}{\sigma}.$$

In fact, we have:

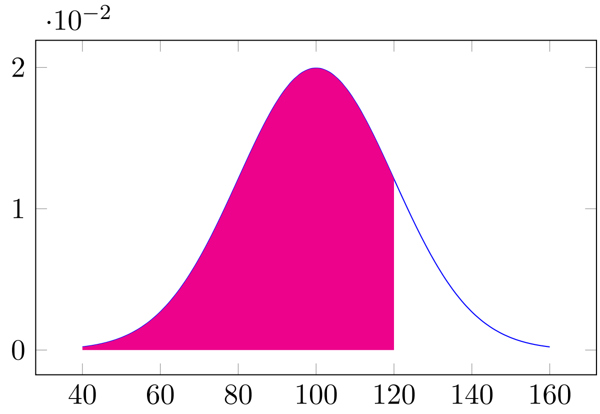
$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right).$$

Example 8.45. Let X be a normal variable with $\mu = 100$ and $\sigma = 20$. Find the values of:

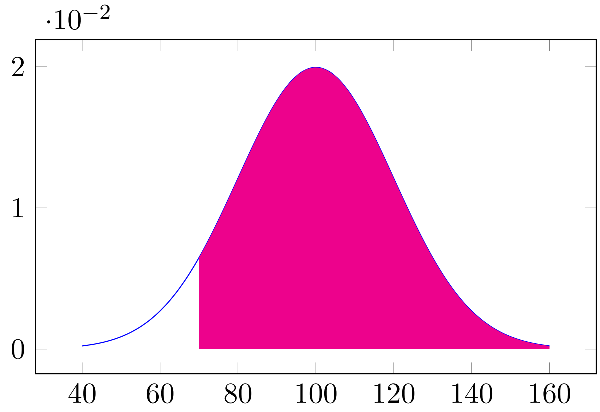
1. $P(X < 120)$.
2. $P(X > 70)$.
3. $P(75 < X < 110)$.

Solution: We have:

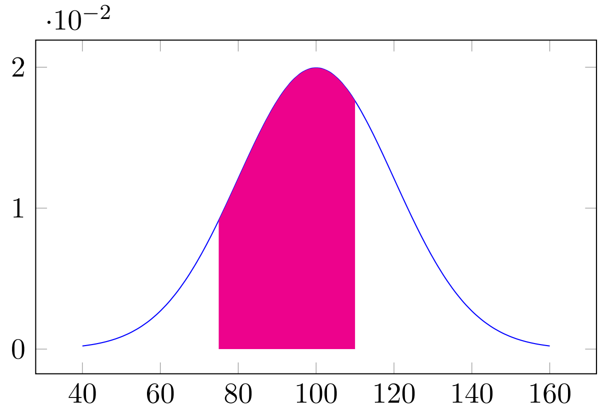
1. $P(X < 120) \approx 0.8413$ and:



2. $P(X > 70) \approx 0.9332$ and:



3. $P(75 < X < 110) \approx 0.5859$ and:



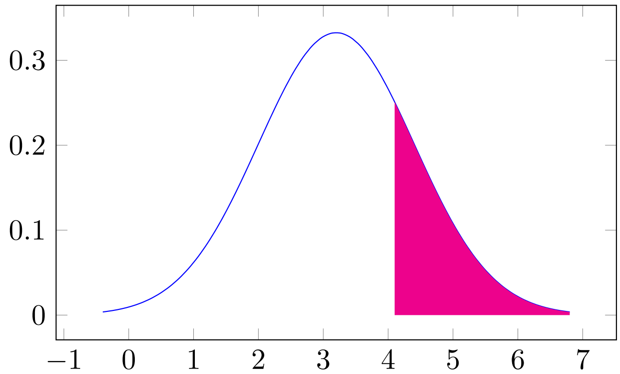
8.6 Applications of the Normal Distribution

Example 8.46. The medical records of infants delivered at the Houston Methodist Hospital show that the infants' weight in kilograms are normally distributed with a

mean of $3.2kg$ and a standard deviation of 1.2. Find the probability that an infant selected at random from among those delivered at the hospital weighed more than $4.1kg$ at birth.
Solution: We have that:

$$P(X > 4.1) = P(Z > 0.75) = 0.2266$$

and:



Example 8.47. The company Hill Country Fare ships potatoes to its distributors in bags whose weights are normally distributed with a mean weight of $50kg$ and standard deviation of $0.5kg$. If a bag of potatoes is selected at random from a shipment, what is the probability that it weights:

1. More than $51kg$.
2. Less than $59kg$.
3. Between 49 and 51 kilograms.

Solution: We have that:

1. $P(X > 51) = 0.02275$.
2. $P(X < 49) = 0.02275$.
3. $P(49 < X < 51) = 0.9545$.

Example 8.48. The GPA of the senior class of Texas A&M Consolidated High School is normally distributed with a mean of 2.7 and a standard deviation of 0.4 points. The seniors in the top 10% of the class are eligible for admission to any of the Texas A&M University campuses. What is the minimum GPA that a senior should have to ensure their eligibility for admission?
Solution: We have X with $\mu = 2.7$ and $\sigma = 0.4$. We want to find x such that:

$$0.1 = P(X \geq x) = P(Z \geq (x - 2.7)/0.4) = P(Z \geq 2.5x - 6.75) = P(Z \leq -2.5x + 6.7)$$

and since $P(Z \leq -1.28) = 0.1003$ we have $-2.5x + 6.7 \approx -1.28$ and thus we have $x \approx 3.213$.

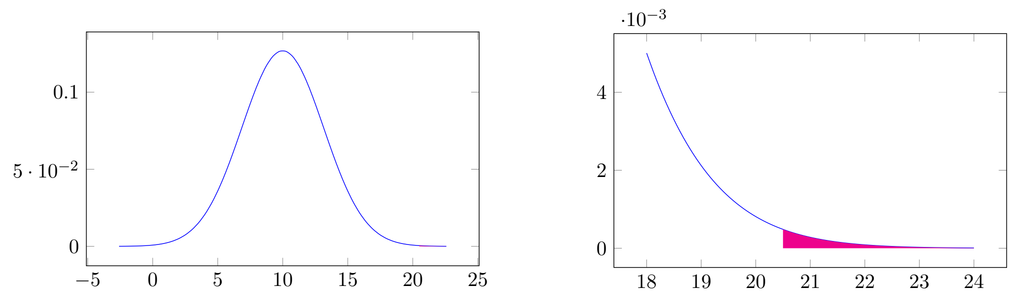
The normal distribution may be used to accurately approximate other distributions. For example, a binomial distribution may be approximated by a normal distribution.

Theorem 8.49. *Let X be a binomial random variable associated to a binomial experiment involving n trials, each with probability of success p and $q = 1 - p$. Then if n is large and p is not close to 0 nor 1, the binomial distribution associated to X may be approximated by a normal distribution having $\mu = np$ and $\sigma = \sqrt{npq}$.*

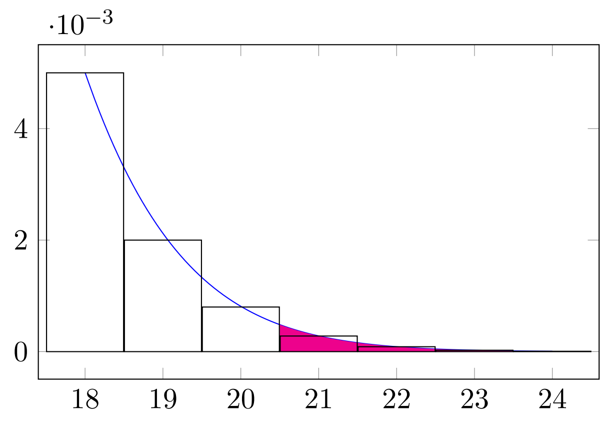
Example 8.50. General Motors receives the microprocessors used to regulate fuel consumption in its automobiles in shipments of 1000 each. It has been estimated that, on average, 1% of the microprocessors that General Motors receives are defective. Determine the probability that more than 20 of the microprocessors in a single shipment are defective.
Solution: We have that finding a defective microprocessor is a binomial experiment, in this case with $n = 1000$ and $p = 0.01$. This means that we can approximate it with a normal distribution Y with $\mu = 10$ and $\sigma = 3.1464$. Hence:

$$P(Y > 20) \approx P(Y > 20.5) \approx P(Z > 3.33) = P(Z < -3.33) = 0.0004$$

with graphical representation:



which indeed approximates the binomial:

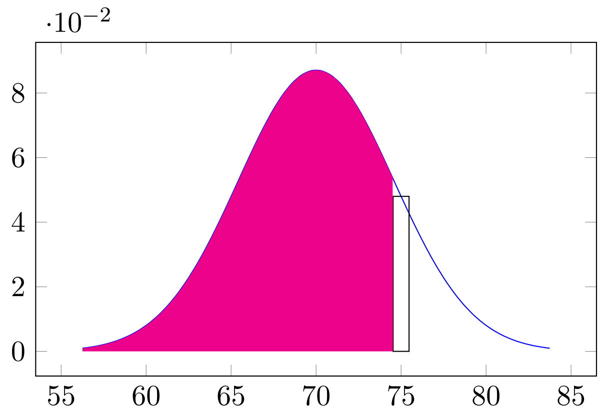


Example 8.51. The probability that a patient with a heart transplant performed at the Baylor Transplant Clinic survives for a year or more after the surgery is 0.7. Of 100 patients who have undergone a heart transplant, what is the probability that:

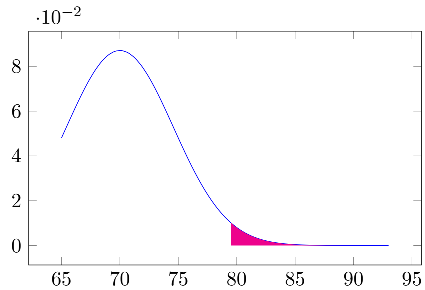
1. Fewer than 75 will survive a year or more after the surgery.
2. Between 80 and 90, inclusive, will survive a year or more after the operation.

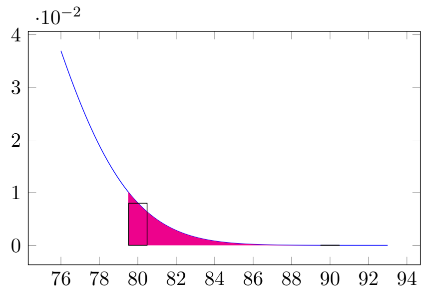
Solution: We have that surviving a year or more after the surgery is a binomial experiment, in this case with $n = 100$ and $p = 0.7$. This means that we can approximate it with a normal distribution Y with $\mu = 70$ and $\sigma = 4.5826$. Hence:

1. $P(Y < 75) \approx P(Y < 74.5) \approx P(Z < 0.98) = 0.8365$, with graphical representation:



2. $P(80 < Y < 90) \approx P(79.5 < Y < 90.5) \approx P(2.07 < Z < 4.48) = 0.0192$, with graphical representation:





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[1] Soo T. Tan, “Finite Mathematics for the Managerial, Life, and Social Sciences” (11th edition), 2014.

[2] `https://xkcd.com`

[3] `https://mathwithbaddrawings.com`