Math 31B
Integration and Infinite Series

Final

Instructions: You have 3 hours to complete this exam. There are 24 questions, worth a total of 24 points. This test is closed book and closed notes. No calculator is allowed. Please clearly box your final answer. Do not forget to write your name, section, and UID in the space below. Once the 3 hours have elapsed, you are not allowed to continue writing and you are not allowed to communicate with anybody except the administrators of the exam. Please follow their requests at all times. Failure to comply with any of these instructions may have repercussions in your final grade.

Name: ______________________________

ID number: ______________________________

Section: ______________________________

Question	Points	Score
1	1	
2	1	
3	1	
4	1	
5	1	
6	1	
7	1	
8	1	
9	1	
10	1	
11	1	
12	1	
13	1	
14	1	
15	1	
16	1	
17	1	
18	1	
19	1	
20	1	
21	1	
22	1	
23	1	
24	1	
Total:	24	

Problem 1. *1pts.*

Find the derivative of $f(x) = \frac{e^{x^2}}{x}$.

Solution: The derivative is $f'(x) = \frac{2x^2e^{x^2} - e^{x^2}}{x^2}$.

Problem 2. *1pts.*

Find the critical points, the points of inflection, and sketch the graph of $f(x) = xe^{-x}$.

Solution: The critical point is $x = 1$, the point of inflection is $x = 2$. The function is increasing for $x < 1$, decreasing for $x > 1$, concave down for $x < 2$, and concave up for $x > 2$.

Problem 3. *1pts.*

Find a domain where $f(x) = \frac{1}{\sqrt{x^2+1}}$ is one-to-one, and find a formula for the inverse restricted to this domain. Sketch the graph of f and f^{-1}.

Solution: The inverse is $f^{-1}(x) = \frac{\sqrt{1-x^2}}{x}$ with domain $x > 0$. The inverse is $f^{-1}(x) = -\frac{\sqrt{1-x^2}}{x}$ with domain $x < 0$.

Problem 4. *1pts.*

Find the derivative of $f(x) = e^{(\ln(x))^2}$.

Solution: The derivative is $f'(x) = e^{(\ln(x))^2} \cdot \frac{2\ln(x)}{x}$.

Problem 5. *1pts.*

Evaluate $\lim_{x\to 1} \frac{e^x - e}{\ln(x)}$.

Solution: The limit is e.

Problem 6. *1pts.*

Evaluate $\int \frac{\arctan(x)dx}{1+x^2}$.

Solution: The integral is $\frac{(\arctan(x))^2}{2} + C$.

Problem 7. *1pts.*

Evaluate $\int \frac{dx}{1+9x^2}$ in terms of inverse hyperbolic functions.

Solution: The integral is $\frac{\arctan(3x)}{3} + C$.

Problem 8. *1pts.*

Evaluate $\int \frac{x^3 dx}{\sqrt{4-x^2}}$.

Solution: The integral is $-\frac{1}{3}\sqrt{4-x^2}(x^2+8)+C$.

Problem 9. *1pts.*

Evaluate $\int \frac{100x dx}{(x-3)(x^2+1)^2}$.

Solution: The integral is $3\ln|x-3| - \frac{3}{2}\ln|x^2+1| - 4\arctan(x) + \frac{5x+15}{x^2+1} + C$.

Problem 10. *1pts.*

Evaluate $\int_1^\infty \frac{\ln(x)dx}{x^2}$.

Solution: The integral is 1.

Problem 11. *1pts.*

Compute the volume of the solid obtained by rotating the region below the graph of $f(x) = e^{-|x|/2}$ about the x-axis for $-\infty < x < +\infty$.

Solution: The volume is 2π.

Problem 12. *1pts.*

Calculate the arc length of $f(x) = \ln(\cos(x))$ over the interval $[0, \pi/4]$.

Solution: The arc length is $\ln(\sqrt{2} + 1)$.

Problem 13. *1pts.*

Use the error bound to find the smallest value of n for which $|\ln(1.3) - T_n(1.3)| \leq 10^{-4}$, centered at $a = 1$.

Solution: The integer is $n = 6$.

Problem 14. *1pts.*

Determine the limit of the sequence $a_n = \frac{e^n}{2^n}$, or explain why it diverges.

Solution: The sequence diverges.

Problem 15. *1pts.*

Determine the limit of the sequence $a_n = \sqrt{n} \cdot \ln\left(1 + \frac{1}{n}\right)$, or explain why it diverges.

Solution: The sequence converges to 0.

Problem 16. *1pts.*

Determine the sum of the series $\sum_{n=1}^{\infty} \frac{n}{10n+12}$, or explain why it diverges.

Solution: The series diverges.

Problem 17. *1pts.*

The Koch snowflake. Step 0 is an equilateral triangle with side of length L and area A_0. Produce stage 1 by dividing each edge in thirds, glue an equilateral triangle of lenght $L/3$ to each edge, and remove the base of each triangle added. Continue the process: At the n-th step, replace each edge with four edges of one-third the length, the two ones in the middle forming a baseless equilateral triangle. Find a recursive formula for the perimeter P_n of the polygon at the n-th step. Rewrite P_n as a geometric series and prove that $\lim_{n\to\infty} P_n = \infty$. Find a formula for T_n the number of triangles added at the n-th step. Find a recursive formula for A_n the area each single triangle added at the n-th step. Rewrite A_n as a geometric series. Compute A the total area of the Koch snowflake in terms of A_0.

Solution: We replace each edge by four edges, so the perimeter at the n-th iteration is 4/3 the perimeter at the $(n-1)$-th iteration, so $P_n = \frac{4}{3}P_{n-1}$. Then $P_n = P_0\left(\frac{4}{3}\right)^n$ so $\lim_{n\to\infty} = \infty$. Each replacement of the edges adds 4 edges, so at the $(n-1)$-th iteration we have $3\cdot 4^{n-1}$ edges. Moreover each replacement of the edges generates a new triangle, so at the n-th iteration we generate $3\cdot 4^{n-1}$ triangles. Also each edge at the n-th iteration has has length 1/3 of the length of the edges at the n-th iteration, so each triangle added at the n-th iteration has area 1/9 of the area of the triangles added at the $(n-1)$-th iteration. The total new area added is then the number of triangles added at the n-th iteration, which is $T_n = 3\cdot 4^{n-1}$ the number of edges at the $(n-1)$-th iteration, multiplied by the area of each new triangle added, which is $A_n = \frac{A_{n-1}}{9} = \frac{A_0}{9^n}$ the area of the new triangles added at the $(n-1)$-th iteration divided by 9. Setting A_0 the area of the original triangle, we obtain $A = A_0 + \sum_{n=1}^{\infty} T_n \cdot A_n = \frac{8A_0}{5}$.

Problem 18. *1pts.*

Determine whether the sum $\sum_{n=1}^{\infty} \frac{4}{n!+4^n}$ converges or diverges, and explain why.

Solution: The sum converges.

Problem 19. *1pts.*

Determine whether the sum $\sum_{n=1}^{\infty} \frac{(\ln(n))^{12}}{n^{9/8}}$ converges or diverges, and explain why.

Solution: The sum converges.

Problem 20. *1pts.*

Determine whether the sum $\sum_{n=1}^{\infty}(-1)^n n^2 e^{-n^3/3}$ converges or diverges, and explain why.

Solution: The sum converges absolutely.

Problem 21. *1pts.*

Determine whether the sum $\sum_{n=1}^{\infty} \frac{n^2+4n}{3n^4+9}$ converges or diverges, and explain why.

Solution: The sum converges.

Problem 22. *1pts.*

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{8^n}{n!} x^n$.

Solution: The interval of convergence is $(-\infty, +\infty)$.

Problem 23. *1pts.*

Find the interval of convergence of the power series $\sum_{n=12}^{\infty} e^n(x-2)^n$.

Solution: The interval of convergence is $(2-\frac{1}{e}, 2+\frac{1}{e})$.

Problem 24. *1pts.*

Find the terms through degree four of the Maclaurin series of $f(x) = e^x \arctan(x)$.

Solution: The terms are $x + x^2 + \frac{x^3}{6} - \frac{x^4}{6}$.