S. 11.1.:

Limit laws for sequences:

Assume that $\{a_n\}$ and $\{b_n\}$ are converging sequences, $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$. Then:

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M$$

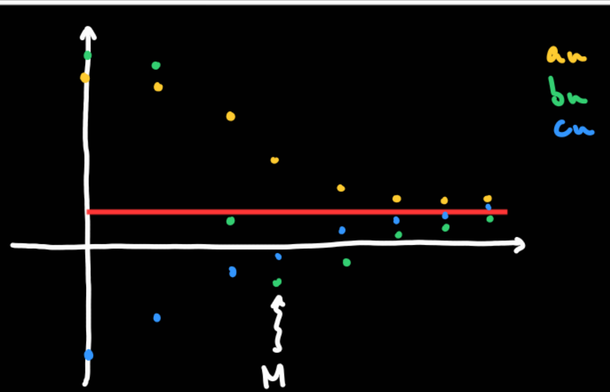
$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M} \quad \text{whenever } M \neq 0.$$

$$\lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n = c \cdot M \quad \text{for any constant } c.$$

Squeeze theorem for sequences:

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that for some M, we have that for $n > M$ then $a_n \leq c_n \leq b_n$. If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$, then $\lim_{n\to\infty} c_n = L$



Example: Show that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

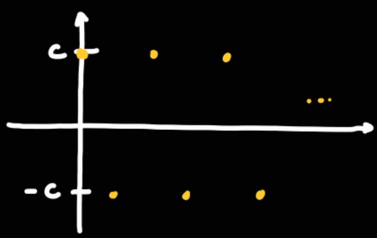
We have $-|a_n| \le a_n \le |a_n|$, and we have $\lim\limits_{n\to\infty} -|a_n| = -\lim\limits_{n\to\infty} |a_n| = 0$. So by the squeeze theorem, $\lim\limits_{n\to\infty} a_n = 0$.

<u>Example:</u> Compute for $r<0$ and $c \neq 0$.

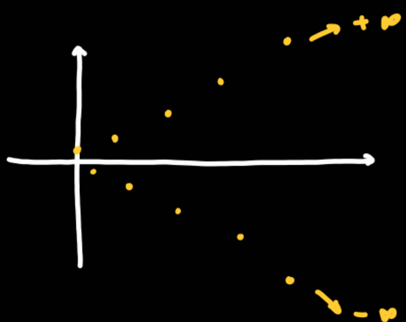
$$\lim_{n\to\infty} c\cdot r^n = \begin{cases} 0 & \text{if } -1<r<0. \\ \text{diverge} & \text{if } r \le -1. \end{cases}$$

If $-1<r<0$, then $0<|r|<1$, and $\lim\limits_{n\to\infty} |c r^n| = \lim\limits_{n\to\infty} |c|\cdot|r^n| = 0$. Thus $\lim\limits_{n\to\infty} c\cdot r^n = 0$ by the Example above.

If $r=-1$ then the sequence $a_n = (-1)^n c$ alternates in sign and does <u>not</u> converge.



If $r<-1$ then $a_n = c r^n$ alternates in sign and $|a_n| = |c\cdot r^n|$ grows arbitarily large, it does <u>not</u> have a limit.



<u>Example:</u> Compute $\lim\limits_{n\to\infty} \frac{R^n}{n!} = 0$ for all R. $\left(f(x) = \frac{R^x}{x!},\ n! = n\cdot(n-1)\cdots 2\cdot 1 \right)$

$\pi!$?

We can bring a limit inside a continuous function:

$$\boxed{\text{If } f(x) \text{ is continuous and } \lim_{n\to\infty} a_n = L, \text{ then } \lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).}$$

Example: Compute $\lim_{n\to\infty} f(a_n)$ for $a_n = \frac{3n}{n+1}$ and $f(x) = x^2$.

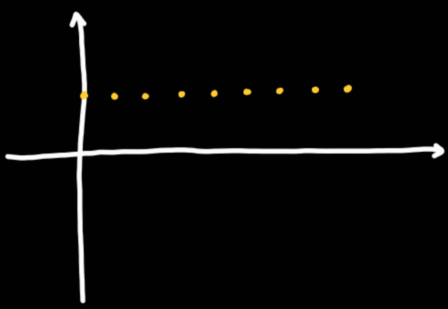
$$\lim_{n\to\infty} f\left(\frac{3n}{n+1}\right) = \lim_{n\to\infty}\left(\frac{3n}{n+1}\right)^2 = \lim_{n\to\infty}\frac{9n^2}{(n+1)^2} = \lim_{n\to\infty}\frac{9n^2}{n^2+2n+1} = 9$$

$$g(x) = \frac{9x^2}{x^2+2x+1}$$

$$\lim_{n\to\infty} f\left(\frac{3n}{n+1}\right) = f\left(\lim_{n\to\infty}\frac{3n}{n+1}\right) = f(3) = 3^2 = 9.$$

$\sin(x)$, $\{1,1,1,\dots\}$ $a_n = 1$

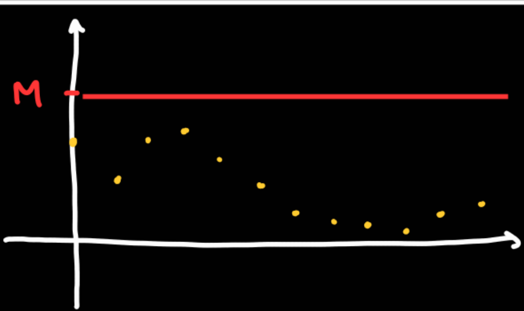
$\{\sin(1), \sin(1), \sin(1), \dots\}$

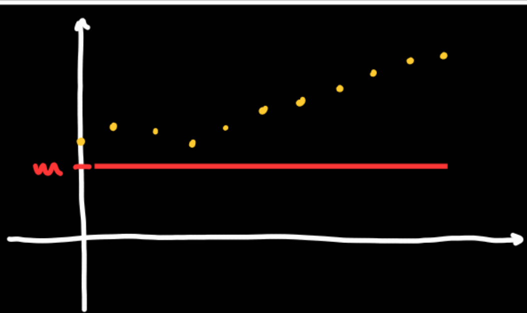


Bounded sequences: A sequence $\{a_n\}$ is:

Bounded from above if there is a number M such that $a_n \leq M$ for all a_n. This number M is called an upper bound.

Bounded from below if there is a number m such that $a_n \geq m$ for all a_n. The number m is called a lower bound.





Convergent sequences are bounded: $\boxed{\text{If } \{a_n\} \text{ converges, then } \{a_n\} \text{ is bounded.}}$

Bounded monotonic sequences converge: (monotonic: always increasing or decreasing)

$$\boxed{\begin{array}{l} \text{If } \{a_n\} \text{ is increasing and } a_n \le M, \text{ then } \{a_n\} \text{ converges and } \lim\limits_{n\to\infty} a_n \le M. \\[2em] \text{If } \{a_n\} \text{ is decreasing and } a_n \ge m, \text{ then } \{a_n\} \text{ converges and } \lim\limits_{n\to\infty} a_n \ge m. \end{array}}$$

Example: Does $\lim\limits_{n\to\infty} \sqrt{n+1} - \sqrt{n}$ exist?

Consider $f(x) = \sqrt{x+1} - \sqrt{x}$, now: $f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} < 0$ for $x > 0$. So $f(x)$ is decreasing, so $a_n = f(n)$ is also decreasing. Also $a_n = \sqrt{n+1} - \sqrt{n} > 0$, so $m = 0$ is a lower bound, so the limit exists.