Section 11.4.: Absolute and conditional convergence.

Absolute convergence: The series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Example: The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges absolutely because $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series.

Absolute convergence implies convergence: If $\sum |a_n|$ converges then $\sum a_n$ converges.

Example: The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges because $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right|$ converges.

Example: The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ does not converge absolutely because $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series.

Conditional convergence:

An infinite series $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

We have that $\sum a_n \leq \sum |a_n|$, so $\sum a_n$ may converge when $\sum |a_n|$ diverges.

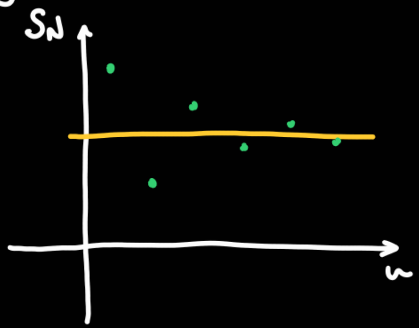
Leibniz test for alternating series:

Assume that $\{a_n\}$ is a positive sequence that is decreasing and converges to zero.

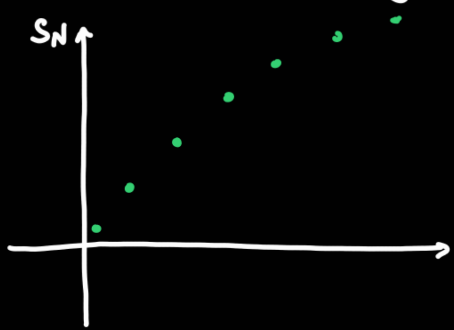
Then the alternating series $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges, $0 < S < a_1$, and

$S_{2N} < S < S_{2N+1}$ for all positive integers N.

<u>Example:</u> The infinite series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges conditionally. The terms $a_n = \frac{1}{\sqrt{n}}$ are positive, and decreasing, and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. Therefore the series converges to some number S by the Leibniz test. Also $0 < S < 1$ since $a_1 = 1$. However the positive series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it is a p-series with $p = \frac{1}{2} < 1$. Therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is convergent but <u>not</u> absolutely convergent, so it is conditionally convergent.



Partial sums of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$.



Partial sums of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Let $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ where $\{a_n\}$ is a positive decreasing sequence that converges to zero. Then $|S - S_N| < a_{N+1}$.

The error commited when approximating S by S_N is less than the first omitted term a_{N+1}.

<u>Example:</u> The alternating harmonic series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally. The terms

$a_n = \frac{1}{n}$ are positive decreasing and $\lim_{n\to\infty} a_n = 0$. So by the Leibniz test, S converges.

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right|$ diverges, so S is conditionally convergent,

but not absolutely convergent. Now:

$|S - S_N| < \frac{1}{N+1}$, and if we want to approximate S while making an error less

than 10^{-3} by choosing an appropriate N we consider the inequality:

$\frac{1}{N+1} \leq 10^{-3}$. Solving for N we find $N+1 \geq 10^3$ so $N \geq 999$.

Finally, we can check:

$$|S - S_{999}| < \frac{1}{999+1} = \frac{1}{1000} = 10^{-3},$$ the desired error bound.

Only for illustrative purposes, if we compute with a calculator $S_{999} \approx 0.69365$.

We will see in Section 11.7 that $S = \ln(2) \approx 0.69314$. Then:

$$|S - S_{999}| \approx |\ln(2) - 0.69365| \approx 0.0005 = \frac{1}{2000} < \frac{1}{1000} = 10^{-3}.$$