Geometric series: Recall that for $|r|<1$ we have $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. Hence:

> $\sum_{n=0}^{\infty} x^n$ is a power series with radius of convergence $R=1$.
>
> Also: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$.

Example: Compute $\sum_{n=0}^{\infty} 2^n \cdot x^n$ for $|x|<\frac{1}{2}$. For this, we substitute $2x$ for x in the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$. We obtain:

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n \cdot x^n,$$

and since the original series is valid for $|x|<1$, the modified series is valid $|2x|<1$, namely $|x|<\frac{1}{2}$.

Example: Find the power series expansion of $\frac{1}{2+x^2}$ with center $c=0$, and find the interval of convergence. We can rewrite:

$$\frac{1}{2+x^2} = \frac{1}{2}\cdot\left(\frac{1}{1+\frac{x^2}{2}}\right) = \frac{1}{2}\cdot\left(\frac{1}{1-\left(\frac{-x^2}{2}\right)}\right) = \frac{1}{2}\cdot\left(\frac{1}{1-u}\right)$$

$$u = \frac{-x^2}{2}$$

and substitute this into a geometric series:

$$\frac{1}{2+x^2} = \frac{1}{2}\left(\frac{1}{1-u}\right) = \frac{1}{2}\cdot\sum_{n=0}^{\infty} u^n = \frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{-x^2}{2}\right)^n = \frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n}}{2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n}}{2^{n+1}}$$

$$u = \frac{-x^2}{2}$$

for $|u|<1$, namely $\left|\frac{-x^2}{2}\right|<1$, so $|x^2|<2$, so $|x|<\sqrt{2}$.

Thus $\frac{1}{2+x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n}}{2^{n+1}}$ for $|x|<\sqrt{2}$, so the interval of convergence is $(-\sqrt{2},\sqrt{2})$.

Term-by-term differentiation and integration:

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Then $F(x)$ is differentiable on $(c-R, c+R)$ if $R<\infty$, and for all x if $R=\infty$. We can integrate and differentiate term by term, so for x in $(c-R, c+R)$:

$$F'(x) = \sum_{n=1}^{\infty} n \cdot a_n \cdot (x-c)^{n-1}$$

$$\int F(x)\,dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1} \cdot (x-c)^{n+1}, \quad A \text{ some constant.}$$

These series have the same radius of convergence R.

<u>Example:</u> Compute $\frac{1}{(1-x)^2}$ as a power series. To do this, we differentiate the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots, \text{ which has radius of convergence } R=1.$$

By differentiating term by term for $|x|<1$:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) = \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) + \frac{d}{dx}(x^4) + \cdots$$

which results in:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots, \text{ which converges for } |x|<1, \text{ namely } -1<x<1.$$

<u>Example:</u> Compute $\arctan(x)$ as a power series. We would like to do this by integrating the geometric series. Note that $\int \frac{1}{1-x} \neq \arctan(x)$, so we need to be a bit more careful. However, we know that $\arctan(x)$ is the antiderivative of $\frac{1}{1+x^2}$. To find a power series for $\frac{1}{1+x^2}$ we substitute $-x^2$ in the usual geometric series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$, obtaining

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$. Since the geometric series is valid for $|x|<1$, the new series is valid for $|-x^2|<1$, namely $|x|<1$. Now, integrating term by term:

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + \cdots)\,dx =$$

$$= \int dx - \int x^2 dx + \int x^4 dx - \int x^6 dx + \cdots = A + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

which converges for $|x|<1$. Since $|0|<1$, $0 = \arctan(0) = A$, and we have:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for } -1<x<1.$$