Problem 7.1.14.:

$$\lim_{x\to\infty} e^{x-x^2} = \lim_{x\to\infty} \frac{1}{e^{x^2-x}} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0.$$

when $x\to\infty$ in a polynomial, the only term that matters is the one with largest exponent.

Problem 7.1.53.: Sketch $f(x) = x\cdot e^{-x}$.

1. Find critical points.

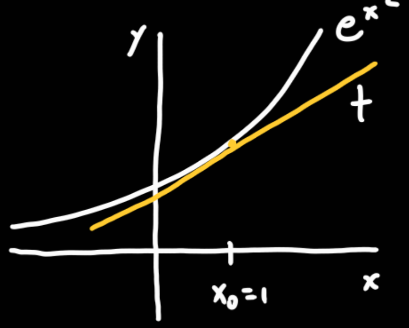
2. See if they are local maximum, minimum, or neither. (p.196)

3. Convexity and concavity.

4. Limits at $x\to\pm\infty$.

$$\lim_{x\to+\infty} x\cdot e^{-x} = \lim_{x\to+\infty} \frac{x}{e^x} = 0$$

Problem 7.1.18: Let $y=e^{x^2}$, $x_0=1$, find the equation of the tangent line.



The slope of t is the derivative of y at $x_0=1$.

This line goes throug $(1,e)$ because it touches $y=e^{x^2}$ at $x_0=1$.

So the equation of t is:

$$y = 2\cdot e\cdot(x-1)+e \quad \text{, i.e.} \quad y-e = 2e(x-1).$$

Problem 7.3.44.:

$$y' = \frac{\cancel{x^3+1}}{x+1}\cdot\frac{(x^3+1)-(x+1)\cdot(3x^2)}{(x^3+1)^{\cancel{2}}} =$$

$$= \frac{x^3+1-(x+1)\cdot 3x^2}{(x+1)(x^3+1)} = \frac{x^3+1-3x^3-3x^2}{(x+1)(x^3+1)} =$$

$$= \frac{-2x^3-3x^2+1}{(x+1)(x^3+1)} =$$

$$\begin{array}{rl|l}
-2x^3-3x^2 & +1 & x+1 \\ \cline{3-3}
-2x^3-2x^2 & & -2x^2+x \quad Q \\ \cline{1-1}
0 \quad x^2 & +1 & \\
x^2 & +x & \\ \cline{1-2}
0 \quad -x & +1 \quad R &
\end{array}$$

$$-2x^3-3x^2+1 = (-2x^2+x)\cdot(x+1)+(-x+1)$$

Problem 7.1.28:

$$f'(x) = 4\cdot(2\cdot e^{3x}+2e^{-2x})^3\cdot(6e^{3x}-4e^{-2x}) =$$

$$= 64\cdot(e^{3x}+e^{-2x})^3\cdot(3\cdot e^{3x}-2\cdot e^{-2x}) = \cdots =$$

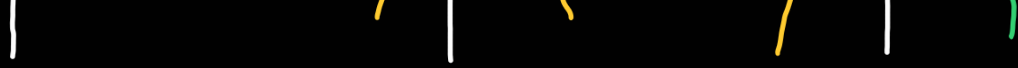
= something with only 5 terms (probably).

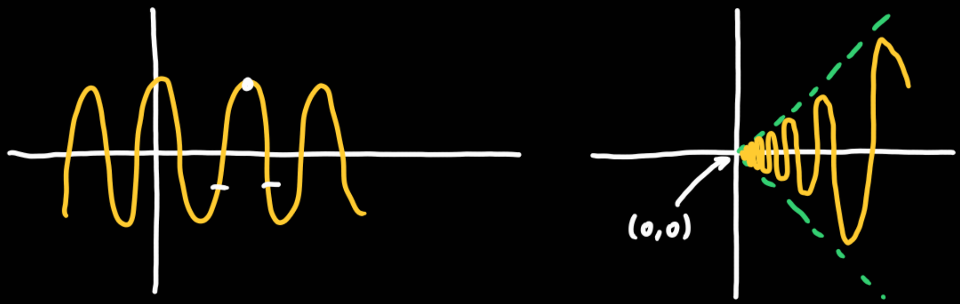
Problem 7.1.51: Critical points of $g(t) = \dfrac{e^t}{t^2+1}$.

$$g'(t) = \frac{(t^2+1)\cdot e^t - 2\cdot e^t\cdot t}{(t^2+1)^2}$$

Critical point: $t=1$. For $t \neq 1$ we have $(t^2+1)^2 > 0$, and

$(t^2+1)\cdot e^t - 2\cdot e^t\cdot t$ is also positive. So $t=1$ is neither a max. nor min.





Problem 7.1.59: Find $a>0$ such that the tangent line to $f(x)=x^2 \cdot e^{-x}$ at $x=a$ passes through the origin.

How to: use point-slope equation of a line with slope $f'(a)$ and point $(0,0)$.

Example: L'Hôpital's rule can fail if not correctly applied.

$$\lim_{x\to 1} \frac{x^2+1}{2x+1} = \frac{1^2+1}{2\cdot 1+1} = \frac{2}{3}.$$

$$\lim_{x\to 1} \frac{x^2+1}{2x+1} = \lim_{x\to 1} \frac{2x}{2} = \frac{2}{2} = 1.$$

⚠ Unlawful application of LHR.

Proof of LHR:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \underset{f(a)=0=g(a)}{=} \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to} \frac{g(x)-g(a)}{x-a}} = \frac{f(a)}{g(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Problem 7.3.75: Differentiate $f(x)=x^{e^x}$. We can use the substitution $u=e^x$.

Problem 7.3.68: Use logarithmic differentiation for $f(x)=\dfrac{x\cdot(x+1)^3}{(3x-1)^2}$.

Simplify the denominator.

$$\frac{y'}{y} = \frac{1}{x} + \frac{3}{x+1} - \frac{6}{3x-1} \quad \text{so} \quad y' = \left(\frac{1}{x} + \frac{3}{x+1} - \frac{6}{3x-1}\right)\cdot y = \cdots \text{ in terms of } x.$$

Problem 7.1.58: Draw $f(x) = x^2 \cdot e^{-x}$ on $[0,10]$.

Problem 7.1.44: $y = \ln\left(\frac{x+1}{x^3+1}\right) = \ln(x+1) - \ln(x^3+1)$

$$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}.$$

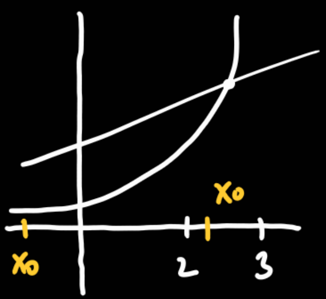
$$(x+1)\cdot(x^2-x+1) = x^3+1$$

$$\begin{array}{l|l}
x^3+1 & x+1 \\ \hline
x^3+x^2 & x^2-x+1 \quad Q \\
\hline
0-x^2-1 & \\
\quad -x^2-x & \\
\hline
\quad 0+x+1 & \\
\qquad x+1 & \\
\hline
\qquad 0 \quad R &
\end{array}$$

$$\frac{x^3+1}{x+1} = x^2-x+1.$$

Newton's method: used to approximate solutions.

Problem 7.1.60: Solve $e^x = 5x$. Consider $y = 5x$ and $y = e^x$.



Consider $f(x) = e^x - 5x$, take $f'(x) = e^x - 5$. Pick x_0.

Then the sequence:

Newton's method $\longrightarrow$ $x_0, \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \ldots, x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$

converges to the solution of $f(x) = 0$.

Problem 7.4.38:

a) $F(t) = e^{-kt}$, inverse $t(F) = \frac{\ln(F)}{-k}$.

b) Riemann sum: how integrals are defined.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} t\left(\frac{j}{N}\right) =: \int_0^1 t(F)\, dF.$$

c) $\frac{d}{dx}\big(\underbrace{x\cdot\ln(x) - x}_{x}\big) = \ln(x)$

$$M = \lim_{c\to 0} \int_c^1 t(F)\,dF = \lim_{c\to 0}\left(\frac{-1}{k}\cdot \overbrace{(F\cdot\ln(F) - F)}^{F}\Big|_c^1 \right) = \cdots =$$

by Fundamental Theorem of Calculus

$$= \lim_{c\to 0}\left(\frac{1}{k} + \frac{1}{k}\left(c\cdot\ln(c) - c\right)\right).$$

d) For $g(c) = c\cdot\ln(c) - c$, we have:

c	0.01	0.0001	0.00001
$g(c)$	-0.0560	-0.0010	-0.0001

so $g(c) \xrightarrow[c\to 0]{} 0$. So $M \xrightarrow[c\to 0]{} \frac{1}{k}$.

e) Half-life is 3.825 days, $k = \frac{\ln(2)}{3.825}$ so $\frac{1}{k} = 5.52$ so $M = 5.52$ days.