Problem 7.7.52: $\displaystyle \lim_{x\to 1} \frac{x^m-1}{x^n-1} \overset{\frac{0}{0}\text{ use LHR}}{=} \lim_{x\to 1} \frac{m\cdot x^{m-1}}{n\cdot x^{n-1}} = \lim_{x\to 1} \frac{m}{n}\cdot x^{m-n} = \frac{m}{n}.$

Problem 7.7.65:

a) Compute $G(b) = \lim_{x\to\infty} (1+b^x)^{\frac{1}{x}}$ knowing that $H(b) = \lim_{x\to\infty} \frac{\ln(1+b^x)}{x} = \ln(b)$ for $b \geq 1$ and $\underbrace{H(b) = \lim_{x\to\infty} \frac{\ln(1+b^x)}{x} = 0}_{\text{solution to 7.7.64 (b)}}$ for $0 < b \leq 1$.

Use the same trick as to solve $\lim_{x\to 0} x^x$.

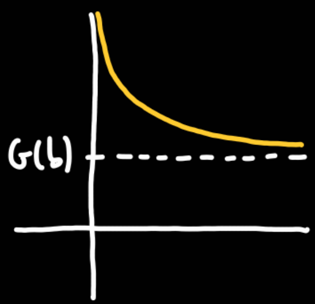
$$G(b) = \lim_{x\to\infty} (1+b^x)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln\left((1+b^x)^{\frac{1}{x}}\right)} = \lim_{x\to\infty} e^{\frac{1}{x}\cdot\ln(1+b^x)} =$$

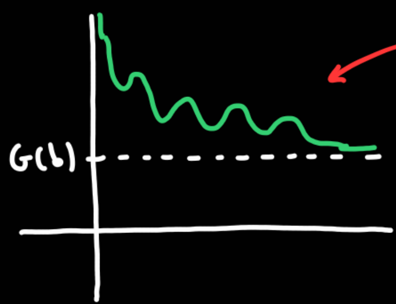
$$= e^{\lim_{x\to\infty} \frac{\ln(1+b^x)}{x}} = e^{H(b)}.$$

If $b \geq 1$ then $H(b) = \ln(b)$ so $G(b) = b$.

If $0 < b \leq 1$ then $H(b) = 0$ so $G(b) = 1$.

b) This is saying that the function $y = (1+b^x)^{\frac{1}{x}}$ goes to $G(b)$ when x is very big. Check that y is decreasing and $\lim_{x\to 0^+} y = +\infty$, so:





we need to check y decreasing!

Problem 7.7.10: $\lim \frac{\cos(x) - \sin^2(x)}{\ldots}$

Problem 7.7.[illegible]: $\lim_{x\to 0} \frac{[illegible]}{\sin(x)}$

plug in $x=0$: $\frac{1-0}{0} = \frac{1}{0}$, it is not one of the indeterminates for LHR. So LHR does not apply.

$$\frac{\cos(x)-\sin^2(x)}{\sin(x)} = \frac{\cos(x)}{\sin(x)} - \frac{\sin^2(x)}{\sin(x)} = \frac{\cos(x)}{\sin(x)} - \sin(x)$$

$$\lim_{x\to 0} \frac{\cos(x)}{\sin(x)} - \sin(x) = \frac{1}{0} - 0$$

Problem 7.7.64: Compute $H(b) = \lim_{x\to\infty} \frac{\ln(1+b^x)}{x}$.

a) For $b \geq 1$:

$$\lim_{x\to\infty} \frac{\ln(1+b^x)}{x} =$$

intuitively, this is $\frac{\ln(b^x)}{x}$, more or less $\frac{x}{x} = 1$.

$\frac{\ln(1+\infty)}{\infty} = \frac{\infty}{\infty}$, apply LHR

(if $b>1$)

$\frac{d}{dx}(x) = 1$

$\frac{d}{dx}\left(\ln(1+b^x)\right) = \frac{\ln(b)\cdot b^x}{1+b^x}$

$$= \lim_{x\to\infty} \frac{\ln(b)\cdot b^x}{1+b^x} = \lim_{x\to\infty} \frac{\ln(b)\cdot \cancel{b^x}}{\cancel{b^x}} = \lim_{x\to\infty} \ln(b) = \ln(b)$$

$y = b^x$ then $1+b^x = 1+y$

if $x\to\infty$ then $y\to\infty$.

whenever a polynomial y^3+4y^2+y has $y\to\infty$ the only term that matters is the highest exponent.

b) For $0<b<1$:

use at your own discretion.

$$\lim_{x\to\infty} \frac{\ln(1+b^x)}{x} = \lim_{x\to\infty} \frac{\ln(b)\cdot b^x}{1+b^x} = \frac{0}{1} = 0.$$

LHR

$b^x \xrightarrow[x\to\infty]{} 0$ because $0<b<1$.

$$\lim \frac{\ln(1+b^x)}{} = \frac{\ln(1+0)}{\infty} = \frac{0}{\infty} = 0.$$ ← use this.

For $b=1$:

$$\lim_{x\to\infty} \frac{\ln(1+b^x)}{x} = \lim_{x\to\infty} \frac{\ln(2)}{x} = 0 = \ln(1) = \ln(b).$$

Hyperbolic functions:

Taking derivatives is easy: we are computing a limit.

Taking integrals is complicated: we are computing inverse functions.

We want to know many pairs $f(x)$ and $g(x)$ such that $f'(x) = g(x)$ because then $\int g(x)\,dx = f(x)$.

For trigonometric functions we have:

$$\frac{d}{dx}\left(\arcsin(x)\right) = \underbrace{\frac{1}{\sqrt{1-x^2}}}, \quad \frac{d}{dx}\left(\arctan(x)\right) = \underbrace{\frac{1}{x^2+1}}.$$

we know how to integrate these. But what if

we want $\int \frac{dx}{\sqrt{1+x^2}}$ and $\int \frac{dx}{1-x^2}$?

We cannot find them with regular trigonometric functions. The reason to look at hyperbolic trigonometric functions is because they will give a solution:

$$\frac{d}{dx}\left(\operatorname{arcsinh}(x)\right) = \frac{1}{\sqrt{1+x^2}}, \quad \frac{d}{dx}\left(\operatorname{arctanh}(x)\right) = \frac{1}{1-x^2}.$$

This is why we want them.

How do we find them? Note that $\sin(\theta)$ and $\cos(\theta)$ satisfy $x^2+y^2=1$.

We will want $\sinh(\theta)$ and $\cosh(\theta)$ to satisfy $x^2-y^2=1$.

Problem 7.8.57:

$$\int \frac{dx}{\underbrace{9+x^2}_{\text{resembles } \frac{1}{1+x^2}}} = \int \frac{dx}{9\cdot\left(1+\frac{x^2}{9}\right)} = \int \frac{dx}{9\cdot\left(1+\left(\frac{x}{3}\right)^2\right)} \overset{u=\frac{x}{3} \quad du=\frac{dx}{3}}{=} \int \frac{du}{3\cdot(1+u^2)} =$$

$$\overset{\text{trig. derivatives}}{=} \frac{1}{3}\cdot\arctan(u) + C_1 = \frac{1}{3}\cdot\arctan\left(\frac{x}{3}\right) + C_1.$$