Reduction formula:

$$\int \sin^n(x)\,dx = -\cos(x)\cdot\sin^{(n-1)}(x) + \int \cos(x)\cdot(n-1)\cdot\sin^{(n-2)}(x)\cdot\cos(x)\,dx =$$

$u = \sin^{(n-1)}(x)$ $\quad du = (n-1)\cdot\sin^{(n-2)}(x)\cdot\cos(x)\,dx$

$dv = \sin(x)\,dx$ $\quad v = -\cos(x)$

$$\int \underbrace{u\,dv}_{n} = uv - \int \underbrace{v\,du}_{n-1}$$

$$= -\cos(x)\cdot\sin^{(n-1)}(x) + \int \cos^2(x)\cdot(n-1)\cdot\sin^{(n-2)}(x)\,dx =$$

$1 = \cos^2(x) + \sin^2(x)$

$$= -\cos(x)\cdot\sin^{(n-1)}(x) + \int (1-\sin^2(x))\cdot(n-1)\cdot\sin^{(n-2)}(x)\,dx =$$

$$= -\cos(x)\cdot\sin^{(n-1)}(x) + (n-1)\cdot\int \sin^{(n-2)}(x)\,dx - (n-1)\int \sin^{n}(x)\,dx$$

This can be further simplified.

$\sin^n(x) = \sin(x)\overset{n}{\cdots}\sin(x)$ are independent functions, we are free to choose u, dv as we want.

Problem 8.1.18.:

$$\int e^{3x}\cdot\cos(4x)\,dx =$$

$u = \cos(4x)$

$dv = e^{3x}\,dx$

Problem 8.1.26.:

$$\int x\cdot\tan(x)\cdot\sec(x)\,dx = x\cdot\sec(x) - \int \sec(x)\,dx = x\cdot\sec(x) - \ln|\sec(x)+\tan(x)| + C.$$

$u = x$ $\quad du = dx$

$dv = \tan(x)\cdot\sec(x)$ $\quad v = \sec(x)$

$$\int \sec(x)\,dx = \int \sec(x)\cdot\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx = \int \frac{\sec^2(x)+\sec(x)\cdot\tan(x)}{\sec(x)+\tan(x)}\,dx =$$

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$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec(x)+\tan(x)| + C.$$

$u = \sec(x) + \tan(x)$

$du = \sec^2(x) + \sec(x)\cdot\tan(x)$

Problem 8.1.38.:

$$\int x^3\cdot e^{x^2}\,dx = \circledast$$

$u = x^3$
$dv = e^{x^2}$ $\quad v = ?$

$\int e^{x^2}\,dx$ is not an analytic integral.

Instead: $y = x^2,\ dy = 2x$

$\int x \cdot e^{x^2}\,dx = \frac{e^{x^2}}{2}$

$\circledast = \int \frac{1}{2} y \cdot e^y\,dy$

$\int x^n \cdot e^{x^m}\,dx$

with $m < n$, try u-sub.

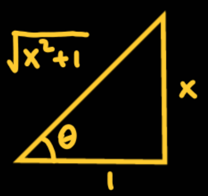
Problem 8.5.40: $\int \frac{100x\,dx}{(x-3)(x^2+1)^2} =$

Partial fraction decomposition:

$$\frac{100x}{(x-3)(x^2+1)^2} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} = \cdots = \underbrace{\frac{3}{x-3}}_{\ln|x-3|} + \underbrace{\frac{-3x-9}{x^2+1}}_{\substack{\ln|x^2+1| \\ \arctan(x)}} + \underbrace{\frac{-30x+10}{(x^2+1)^2}}_{\frac{1}{x^2+1}}$$

$\int \frac{10}{(x^2+1)^2}\,dx$

$$\int \frac{10\,dx}{(x^2+1)^2} \underset{x=\tan(\theta)}{=} \cdots = \int \cos^2(\theta)\,d\theta \underset{\text{integration by parts}}{=} \cdots = \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cdot\cos(\theta) =$$



$\sin(\theta) = \frac{x}{\sqrt{x^2+1}}$

$\cos(\theta) = \frac{1}{\sqrt{x^2+1}}$

$$= \frac{\arctan(x)}{2} + \frac{x}{2(x^2+1)} + C.$$

Claim: $\int \left(\arctan(x) + \frac{x}{1+x^2}\right)dx = x \cdot \arctan(x) + \frac{x^2}{1+x^2} + \cdots$

$$\frac{d}{dx}\left(x \cdot \arctan(x)\right) = \arctan(x) + x \cdot \frac{1}{x^2+1}$$

$$\int \arctan(x) = x \cdot \arctan(x) - \frac{1}{2}\ln|1+x^2|$$

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln|1+x^2|$$

So: $\int \left(\arctan(x) + \frac{x}{1+x^2}\right)dx = \int \arctan(x)\,dx + \int \frac{x\,dx}{1+x^2}$.

$\circledast$ for solution

Integrate: $\int \cos^2(x)\,dx = \cos(x)\cdot\sin(x) + \int \sin^2(x)\,dx = \cos(x)\cdot\sin(x) + \int (1-\cos^2(x))\,dx =$

$u = \cos(x)$ $du = -\sin(x)$

$dv = \cos(x)$ $v = \sin(x)$

$\cos^2(x) + \sin^2(x) = 1$

$$= \cos(x)\cdot\sin(x) + x - \int \cos^2(x)\,dx$$

$$2\cdot\int \cos^2(x)dx = \cos(x)\cdot\sin(x) + x \qquad \text{so :} \quad \int \cos^2(x)dx = \frac{x}{2} + \frac{\cos(x)\cdot\sin(x)}{2} + c.$$

Integrate: $\int \frac{1}{x^2-4}dx = \int\left(\frac{\frac{1}{4}}{\underbrace{x-2}} - \frac{\frac{1}{4}}{\underbrace{x+2}}\right)dx = \underbrace{\frac{1}{4}\cdot\ln|x-2|} - \underbrace{\frac{1}{4}\ln|x+2|}$. for all $x \neq 2$, $x \neq -2$

$x^2-4 = (x-2)(x+2)$ $\quad \frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}$

$1 = A\cdot(x+2) + B\cdot(x-2)$ so $\underset{x=2}{A = \frac{1}{4}}$, $\underset{x=-2.}{B = \frac{-1}{4}}.$

$$\int \frac{1}{x^2-4} \stackrel{?}{=} -\frac{1}{2}\cdot\operatorname{arctanh}\left(\frac{x}{2}\right) + c \qquad \text{only for } x \text{ in } (-1,1)$$

$$\frac{-1}{4}\cdot\frac{d}{dx}\left(\operatorname{arctan}\left(\frac{x}{2}\right)\right) = \frac{\frac{1}{2}}{1-\left(\frac{x}{2}\right)^2}\cdot\frac{-1}{2} = \frac{-1}{4}\cdot\frac{1}{1-\frac{x^2}{4}} = \frac{-1}{4-x^2}$$

⊛ Solution: $\int \frac{100x\,dx}{(x-3)(x^2+1)^2} = 3\cdot\ln|x-3| - \frac{3}{2}\cdot\ln|x^2+1| - 4\cdot\arctan(x) + \frac{5x+15}{x^2+1} + c.$

Integrate by parts:

$$\int \frac{-30x+10}{(x^2+1)^2}dx = (-30x+10)\cdot\left(\frac{x}{2(x^2+1)} + \frac{\arctan(x)}{2}\right) - \int\left(\frac{x}{2(x^2+1)} + \frac{\arctan(x)}{2}\right)(-30\,dx) =$$

$u = -30x + 10$ $\quad du = -30\,dx$

$dv = \frac{1}{(x^2+1)^2}$ $\quad v = \frac{x}{2(x^2+1)} + \frac{\arctan(x)}{2}$

$$\int \frac{dx}{(x^2+1)^2} = \frac{x}{2(x^2+1)} + \frac{\arctan(x)}{2}$$

$u = \frac{1}{(x^2+1)^2}$ $\quad du = \frac{-4x}{(x^2+1)^3}$

(x^2+1) $\quad$ (x^2+1) $\quad$ not good.

$dv = 1 \qquad v = x$

$x = \tan(\theta) \qquad dx = \sec^2(\theta)\,d\theta$

$$= \frac{-15x^2+5x}{x^2+1} - 15x\arctan(x) + 5\arctan(x) + 15\int\left(\frac{x}{x^2+1} + \arctan(x)\right)dx =$$

$$\int\left(\frac{x}{x^2+1} + \arctan(x)\right)dx = \int\frac{x}{x^2+1}\,dx + \int\arctan(x)\,dx =$$

$u = \arctan(x) \qquad du = \frac{1}{x^2+1}$

$dv = 1 \qquad v = x$

$$= \int\frac{x}{x^2+1}\,dx + x\cdot\arctan(x) - \int\frac{x}{x^2+1}\,dx = x\cdot\arctan(x)$$

$$= \frac{-15x^2+5x}{x^2+1} \cancel{-15x\arctan(x)} + 5\arctan(x) \cancel{+15x\arctan(x)} + c_1 =$$

$$= 5\cdot\frac{-3x^2+x}{x^2+1} + 5\arctan(x) + c_1 = 5\cdot\left(\frac{x-3}{x^2+1} + \arctan(x)\right) + \underbrace{5\cdot 3 + c_1}_{\text{constant}} =$$

division of polynomials:

$-3x^2+x = (x^2+1)\cdot 3 + x - 3 \rightsquigarrow \frac{-3x^2+x}{x^2+1} = 3 + \frac{x-3}{x^2+1}$

$$= 5\cdot\left(\frac{x-3}{x^2+1} + \arctan(x)\right) + c_1.$$