Problem 9.1.38: Compute the surface area given by $f(x) = e^{-x}$ in $[0,1]$.

$$S = \int_a^b 2\pi f(x)\cdot \sqrt{1+(f'(x))^2}\,dx \quad , \quad f'(x) = -e^{-x}$$

$$S = 2\pi \int_0^1 e^{-x}\cdot \sqrt{1+e^{-2x}}\,dx =$$

$e^{-x} = \tan\theta \qquad -e^{-x}dx = \sec^2\theta\, d\theta \qquad\qquad 1+\tan^2\theta = \sec^2\theta$

looks like $\sqrt{1+x^2}$, so we want $x = \tan(\theta)$ and then use $\sqrt{1+\tan^2\theta} = \sec\theta$

$$\int e^{-x}\cdot\sqrt{1+e^{-2x}}\,dx = -\int \sec^2\theta\cdot\sec\theta\cdot d\theta = -\int \sec^3\theta\, d\theta = -\left(\sec\theta\cdot\tan\theta - \int \tan^2\theta\,\sec\theta\, d\theta\right) =$$

$e^{-x} = \tan\theta \qquad -e^{-x}dx = \sec^2\theta\, d\theta$

$\sqrt{1+\tan^2\theta} = \sec\theta$

$u = \sec\theta \qquad du = \sec\theta\cdot\tan\theta$

$dv = \sec^2\theta \qquad v = \tan\theta$

$$= -\sec\theta\cdot\tan\theta + \int (\sec^2\theta - 1)\cdot\sec\theta\, d\theta = -\sec\theta\cdot\tan\theta + \int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta$$

$$\int \tan\theta\cdot\sec\theta\cdot\tan\theta\cdot d\theta = \tan\theta\cdot\sec\theta - \int \sec^2\theta\, d\theta$$

$u = \tan\theta \qquad v = \sec^2\theta$

$dv = \tan\theta\sec\theta \qquad v = \sec\theta$

$$-\int \sec^3\theta\, d\theta = -\sec\theta\tan\theta + \int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta$$

$$-2\int \sec^3\theta\, d\theta = -\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta| + C_1$$

$$-\int \sec^3\theta\, d\theta = \frac{-1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C_1 =$$

$e^{-x} = \tan\theta$

$\sqrt{1+e^{-2x}} = \sqrt{1+\tan^2\theta} = \sec\theta$

$$= \frac{-1}{2}e^{-x}\cdot\sqrt{1+e^{-2x}} - \frac{1}{2}\ln\left|\sqrt{1+e^{-2x}} + e^{-x}\right| + C_1$$

$$S = 2\pi\int_0^1 e^{-x}\cdot\sqrt{1+e^{-2x}}\,dx = 2\pi\left(\frac{-1}{2}e^{-x}\cdot\sqrt{1+e^{-2x}} - \frac{1}{2}\ln\left|\sqrt{1+e^{-2x}} + e^{-x}\right|\right)\Bigg|_0^1 =$$

$$= -\pi\cdot e^{-x}\cdot\sqrt{1+e^{-2x}} - \pi\cdot\ln\left|\sqrt{1+e^{-2x}}+e^{-x}\right|\Bigg|_0^1 = -\pi\cdot\frac{1}{e}\cdot\sqrt{1+\frac{1}{e^2}} - \pi\cdot\ln\left(\sqrt{1+\frac{1}{e^2}}+\frac{1}{e}\right)$$

$$+\pi\sqrt{2}+\pi\cdot\ln\left(\sqrt{2}+1\right) = \pi\sqrt{2} - \frac{\pi}{e}\cdot\sqrt{1+\frac{1}{e^2}} + \pi\cdot\ln\left(\frac{\sqrt{2}+1}{\sqrt{1+\frac{1}{e^2}}+\frac{1}{e}}\right).$$

Problem 9.1.37.: Compute the surface area for $f(x) = \left(4-x^{\frac{2}{3}}\right)^{\frac{3}{2}}$ in $[0, 8]$.

$$S = 2\pi\int_0^8 f(x)\cdot\sqrt{1+\left(f'(x)\right)^2}\,dx \qquad f'(x) = \frac{3}{2}\cdot\left(4-x^{\frac{2}{3}}\right)^{\frac{1}{2}}\cdot\left(-\frac{2}{3}\right)\cdot x^{-\frac{1}{3}} = -x^{-\frac{1}{3}}\left(4-x^{\frac{2}{3}}\right)^{\frac{1}{2}}$$

$$1+\left(f'(x)\right)^2 = 1+\frac{4-x^{2/3}}{x^{2/3}} = \frac{x^{\frac{2}{3}}}{x^{\frac{2}{3}}}+\frac{4-x^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{4}{x^{2/3}}$$

$$S = 2\pi\int_0^8\left(4-x^{\frac{2}{3}}\right)^{\frac{3}{2}}\cdot\frac{2}{x^{1/3}}\,dx = 2\pi\int_4^0 u^{\frac{3}{2}}\cdot(-3)\,du = 6\pi\int_0^4 u^{\frac{3}{2}}\,du = 6\pi\left[\frac{2}{5}u^{\frac{5}{2}}\right]\Bigg|_0^4 =$$

$u = 4-x^{\frac{2}{3}}$ $\quad du = -\frac{2}{3}x^{-\frac{1}{3}}dx$

$x=8 \rightsquigarrow u = 4-\left(2^3\right)^{\frac{2}{3}} = 4-4 = 0$

$x=0 \rightsquigarrow u = 4-0 = 4$

$$= 6\pi\cdot\frac{2}{5}\cdot 4^{\frac{5}{2}} = \frac{12\pi}{5}\cdot 2^5 = \frac{12\pi\cdot 32}{5} = \frac{384\pi}{5}$$

Integrate: $\int \csc(x)\,dx = \ln\left|\csc(x)-\cot(x)\right| + C_1$ ☺

$$\int \csc(x)\,dx = -\ln\left|\csc(x)+\cot(x)\right| + C_1$$

$$-\ln\left|\csc(x)+\cot(x)\right| = 0-\ln\left|\csc(x)+\cot(x)\right| = \ln(1)-\ln\left|\csc(x)+\cot(x)\right| =$$

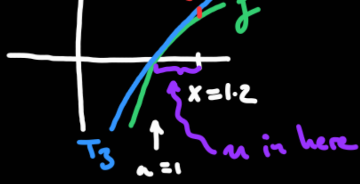
$$= \ln\left|\frac{1}{\csc(x)+\cot(x)}\right| = \ln\left|\frac{1}{\csc(x)+\cot(x)}\cdot\frac{\csc(x)-\cot(x)}{\csc(x)-\cot(x)}\right| =$$

$$= \ln\left|\frac{\csc(x)-\cot(x)}{\csc^2(x)-\cot^2(x)}\right| = \ln\left|\csc(x)-\cot(x)\right|$$

$\sin^2\theta+\cos^2\theta = 1$, $\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$, $1+\cot^2\theta = \csc^2\theta$, $1 = \csc^2\theta-\cot^2\theta$

Example: $f(x) = \ln(x)$, $a=1$, bound the error of $T_3(x)$ at $x=1.2$.

error

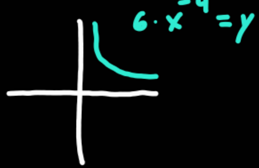


<u>Step 1:</u> $|f(x) - T_3(x)| \leq K \cdot \dfrac{|x-a|^{3+1}}{(3+1)!}$

We know $|f^{(3+1)}(u)| \leq K$ for all u between a and x.

$f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$, $f'''(x) = \frac{2}{x^3}$, $f^{(4)}(x) = \frac{-6}{x^4}$. Now $|f^{(4)}(x)| = 6x^{-4}$ is

decreasing between $a=1$ and $x=1.2$.



So its maximum between $a=1$ and $x=1.2$ is $|f^{(4)}(1)| = 6$, so take $K=6$.

<u>Step 2:</u> Apply the formula.