Problem 11.2.49.: We have that a sequence, $a_n = b_n - b_{n-1}$. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} b_n = L$ exists.

$\sum_{n=1}^{\infty} a_n$ converges whenever $\lim_{N\to\infty} S_N$ is finite.

$$S_N = a_1 + a_2 + a_3 + \cdots + a_{N-1} + a_N =$$

$$= (b_1 - b_0) + (b_2 - b_1) + (b_3 - b_2) + \cdots + (b_{N-1} - b_{N-2}) + (b_N - b_{N-1}) = b_N - b_0.$$

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{N\to\infty} S_N = \lim_{N\to\infty} (b_N - b_0) = \lim_{N\to\infty} (b_N) - b_0$ is finite. So there is a real number L with $\lim_{N\to\infty} (b_N) - b_0 = L$ so $\lim_{N\to\infty} b_N = L + b_0$ exists and is finite.

If $\lim_{n\to\infty} b_n$ exists, then there is some real number L with $\lim_{n\to\infty} b_n = L$. Now $\lim_{N\to\infty} S_N =$ $= \lim_{N\to\infty} (b_N - b_0) = L - b_0$ is finite, so $\sum_{n=1}^{\infty} a_n$ converges.

Problem 11.1.40.: Compute $\lim_{n\to\infty} n^{\frac{1}{n}}$.

Compute $\lim_{x\to\infty} x^x$. We use $e^{\ln(x)} = x$, so $e^{\ln(x^x)} = e^{x\cdot\ln(x)}$. Moreover:

$$\lim_{n\to\infty} e^x = e^{\lim_{n\to\infty} x} \qquad \text{so} \qquad \lim_{x\to\infty} x^x = \lim_{x\to\infty} e^{x\cdot\ln(x)} = e^{\lim_{x\to\infty} x\cdot\ln(x)}.$$

Now: $\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln(n^{\frac{1}{n}})} = \lim_{n\to\infty} e^{\frac{1}{n}\cdot\ln(n)} = e^{\lim_{n\to\infty} \frac{1}{n}\cdot\ln(n)} = \underset{\uparrow\,\text{LHR.}}{\cdots} = 1.$

Problem 11.1.42.: Compute $\lim_{n\to\infty} \frac{8^{2n}}{n!}$. Note $8^{2n} = (8^2)^n = 64^n$

In class: $\lim_{n\to\infty} \frac{R^n}{n!}$.

⚠ If n is a natural number, then $n! = n(n-1)(n-2)\cdots 2\cdot 1$, but if x is a real number, what is $x!$?

We write:

We write:

$$\frac{64^n}{n!} = \frac{64}{1}\cdot\frac{64}{2}\cdots\frac{64}{n-1}\cdot\frac{64}{n} = \underbrace{\frac{64}{1}\cdots\frac{64}{64}}_{c}\cdot\underbrace{\frac{64}{65}\cdots\frac{64}{n}}_{\text{each one is less than 1}} \le c\cdot\frac{64}{n}.$$

Now $0 \le \frac{64^n}{n!} \le c\cdot\frac{64}{n}$ for some c, for $n \ge 64$, so:

$$0 \le \lim_{n\to\infty}\frac{64^n}{n!} \le \lim_{n\to\infty} c\cdot\frac{64}{n} = 0. \quad \text{So} \quad \lim_{n\to\infty}\frac{64^n}{n!} = 0.$$

Problem 11.1.68: Evaluate $\lim_{n\to\infty} c_n$, $c_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$, $\frac{n}{\sqrt{n^2+n}} \le c_n \le \frac{n}{\sqrt{n^2+1}}$.

Sanity check: $n^2+1 \le n^2+m$ for $m = 1, \dots, n$. So $\sqrt{n^2+1} \le \sqrt{n^2+m}$, so

$\frac{1}{\sqrt{n^2+m}} \le \frac{1}{\sqrt{n^2+1}}$ for $m = 1, \dots, n$. So:

$$c_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}}.$$

Similarly $c_n \ge \frac{n}{\sqrt{n^2+n}}$. Now:

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}} = \cdots = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1, \quad \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}} = \cdots = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}} = 1.$$

So by the Squeeze theorem, $\lim_{n\to\infty} c_n = 1$.

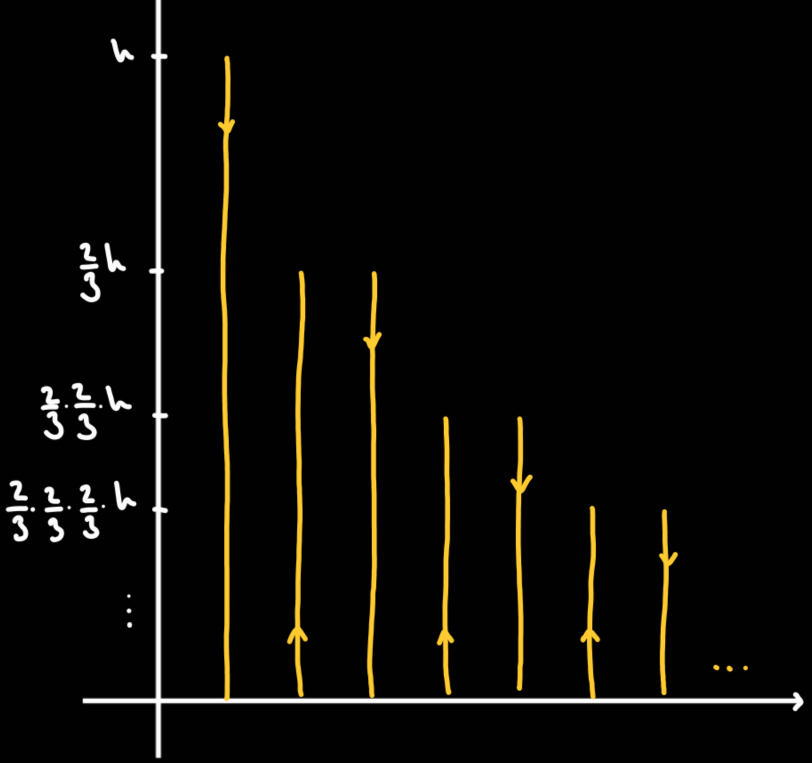
Problem 11.1.64.: $\lim_{n\to\infty}\sqrt{n}\cdot\ln\left(1+\frac{1}{n}\right) = 0$.

Problem 11.2.16.: Find a formula for S_N of $\sum_{n=1}^{\infty}(-1)^{n-1}$ and show it diverges.

$S_1 = 1$, $S_2 = 0$, $S_3 = 1$, $\dots$, $S_N = 1$ if N odd, $S_N = 0$ if N even.

So $\sum_{n=1}^{\infty}(-1)^{n-1} = \lim_{N\to\infty} S_N$ does not converge, so the sum diverges.

Problem 11.2.48.:



$$d = h + 2\cdot\frac{2}{3}\cdot h + 2\cdot\left(\frac{2}{3}\right)^2\cdot h + 2\cdot\left(\frac{2}{3}\right)^3\cdot h + \dots =$$

$$= h + 2\cdot h\cdot \underbrace{\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n}_{\text{geometric series with } r=\frac{2}{3}<1 \text{, so it converges}} = \dots = 5\cdot h \overset{h=10}{=} 50$$

Problem 11.1.62.:

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n = \lim_{n\to\infty} e^{\ln\left(\left(1+\frac{1}{n^2}\right)^n\right)} = \lim_{n\to\infty} e^{n\cdot\ln\left(1+\frac{1}{n^2}\right)} = e^{\lim_{n\to\infty} n\cdot\ln\left(1+\frac{1}{n^2}\right)} = e^0 = 1.$$

$$\lim_{n\to\infty} n\cdot\ln\left(1+\frac{1}{n^2}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n}} \overset{\text{LHR}}{=} \dots = 0$$

Problem 11.2.38.: Show that $\sum_{k=1}^{\infty}\frac{1}{k^{1/3}}$ diverges.

Alternative way: $\int_1^{\infty}\frac{1}{x^{1/3}}dx$ diverges (p-integral with $p<1$), so by the integral test $\sum_{k=1}^{\infty}\frac{1}{k^{1/3}}$ diverges.

The partial sums are:

$$S_N = \frac{1}{1^{1/3}} + \frac{1}{2^{1/3}} + \dots + \frac{1}{N^{1/3}} \geq \underbrace{\frac{1}{N^{1/3}} + \frac{1}{N^{1/3}} + \dots + \frac{1}{N^{1/3}}}_{N} = N\cdot\frac{1}{N^{1/3}} = N^{\frac{2}{3}}.$$

$N \geq m$ then $N^{\frac{1}{3}} \geq m^{1/3}$ so $\frac{1}{m^{1/3}} \geq \frac{1}{N^{1/3}}$

Now:

$$\lim_{N\to\infty} S_N \geq \lim_{N\to\infty} N^{\frac{2}{3}} = \infty \text{, so } \lim_{N\to\infty} S_N = \sum_{k=1}^{\infty}\frac{1}{k^{1/3}} \text{ diverges.}$$

Problem 11.1.35.: $\lim\limits_{n\to\infty} 10+\left(\frac{-1}{9}\right) = \lim\limits_{n\to\infty} 10 + \lim\limits_{n\to\infty}\left(\frac{-1}{9}\right) = 10+0=0.$

if $\lim\limits_{n\to\infty} 10$ converges and $\lim\limits_{n\to\infty}\left(\frac{-1}{9}\right)^n$ converges.

$$\underbrace{-\left(\frac{1}{9}\right)^n}_{\xrightarrow[n\to\infty]{} 0} \le \left(\frac{-1}{9}\right)^n \le \underbrace{\left(\frac{1}{9}\right)^n}_{\xrightarrow[n\to\infty]{} 0}$$

$\lim\limits_{n\to\infty}\left|\frac{-1}{9}\right|^n = \lim\limits_{n\to\infty}\left(\frac{1}{9}\right)^n = 0$, so $\lim\limits_{n\to\infty}\left(\frac{-1}{9}\right)^n = 0$.