Problem 11.3.58.: $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln(n))^{12}}{n^{9/8}}$

We want to compare this with something like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p>1$, so it would converge.

Since $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{n^q} = 0$ for $q>0$, we can use l'Hôpitals to see this. By the definition of the limit, this is saying that $\ln(n) < n^q$ for n big enough (since everything is positive). Namely if $n>M$, then $\ln(n)<n^q$. Choose $q = \frac{1}{12\cdot 16}$, now:

$$\sum_{n=1}^{\infty} \frac{(\ln(n))^{12}}{n^{9/8}} = \sum_{n=1}^{M} \frac{(\ln(n))^{12}}{n^{9/8}} + \sum_{n>M} \frac{(\ln(n))^{12}}{n^{9/8}} \leq \sum_{n=1}^{M} \frac{(\ln(n))^{12}}{n^{9/8}} + \sum_{n>M} \frac{\left(n^{\frac{1}{12\cdot 16}}\right)^{12}}{n^{9/8}} =$$

$$= \sum_{n=1}^{M} \frac{(\ln(n))^{12}}{n^{9/8}} + \sum_{n>M} \frac{n^{\frac{1}{16}}}{n^{9/8}} = \sum_{n=1}^{M} \frac{(\ln(n))^{12}}{n^{9/8}} + \sum_{n>M} \frac{1}{n^{\frac{9}{8}-\frac{1}{16}}} =$$

$$= \sum_{n=1}^{M} \frac{(\ln(n))^{12}}{n^{9/8}} + \sum_{n>M} \frac{1}{n^{17/16}}.$$

which is a converging p-series. Hence the infinite series converges.

Problem 11.3.13.: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{\ln(n)}}$.

Note: $2^{\ln(n)} = \left(e^{\ln(2)}\right)^{\ln(n)} = e^{\ln(2)\cdot\ln(n)} = \left(e^{\ln(n)}\right)^{\ln(2)} = n^{\ln(2)}$.

Now: $\displaystyle\int_1^{\infty} \frac{dx}{x^{\ln(2)}} = \lim_{R\to\infty} \int_1^{R} \frac{dx}{x^{\ln(2)}} = \lim_{R\to\infty} \frac{1}{1-\ln(2)} \cdot x^{1-\ln(2)} \Big|_1^R = \cdots = \infty$, so the series diverges.

Problem 11.3.17.: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$.

It is true that $\frac{1}{n+\sqrt{n}} < \frac{1}{n}$ and $\frac{1}{n+\sqrt{n}} < \frac{1}{\sqrt{n}}$, so:

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} < \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{diverges}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} < \underbrace{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}}_{\text{diverges}}$, but the Comparison test does not help. Instead, we note that $n \geq \sqrt{n}$ so $2n = n+n \geq n+\sqrt{n}$, so $\frac{1}{n+\sqrt{n}} \geq \frac{1}{2n}$. So:

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \cdot \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{diverges}}, \text{ so by the Comparison test } \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} \text{ diverges.}$$

Problem 11.3.30.: $\sum_{n=1}^{\infty} \frac{n!}{n^3}$

Recall that if $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges. We want to compare $\sum_{n=1}^{\infty} \frac{n!}{n^3} \geq \sum b_n$, with $\sum b_n$ divergent. From some point onwards, $n! > n^6$ so $\frac{n!}{n^3} > n^3$. This is because $\lim_{n\to\infty} \frac{n!}{n^6} = \infty$. Namely for $n > M$, then $\frac{n!}{n^3} > n^3$. So:

$$\sum_{n=1}^{\infty} \frac{n!}{n^3} = \sum_{n=1}^{M} \frac{n!}{n^3} + \sum_{n>M} \frac{n!}{n^3} \geq \sum_{n>M} \frac{n!}{n^3} > \sum_{n>M} n^3, \text{ which diverges.}$$

So by the Comparison test, $\sum_{n=1}^{\infty} \frac{n!}{n^3}$ diverges.

✱

Problem 11.3.8.: $\sum_{n=4}^{\infty} \frac{1}{n^2-1}$. Use partial fraction decomposition.

Problem 11.3.41.: $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3+1}}$

Note: $\frac{n}{\sqrt{n^3+1}} \xrightarrow[n\to\infty]{} \frac{n}{\sqrt{n^3}} = \frac{n}{n\sqrt{n}} = \frac{1}{\sqrt{n}}$. By the Limit comparison test:

$$\lim_{n\to\infty} \frac{\frac{n}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \cdots = 1, \text{ and since } \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges, then } \sum_{n=2}^{\infty} \frac{n}{\sqrt{n+1}} \text{ also diverges.}$$

✱

Problem 11.3.8.: $\sum_{n=4}^{\infty} \frac{1}{n^2-1}$.

Using partial fraction decomposition: $\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}$. Integrating:

$$\int_4^{\infty} \frac{dx}{x^2-1} = \int_4^{\infty} \left(\frac{1}{2} \cdot \frac{1}{x-1} - \frac{1}{2} \cdot \frac{1}{x+1}\right) dx = \int_4^{\infty} \frac{1}{2} \cdot \frac{dx}{x-1} - \int_4^{\infty} \frac{1}{2} \cdot \frac{dx}{x+1} =$$

$$= \lim_{R\to\infty} \int_4^{R} \frac{1}{2} \cdot \frac{dx}{x-1} - \lim_{R\to\infty} \int_4^{R} \frac{1}{2} \cdot \frac{dx}{x+1} = \frac{1}{2} \cdot \left(\lim_{R\to\infty} \ln|x-1| - \lim_{R\to\infty} |x+1|\right)\Big|_4^R =$$

$$= \frac{1}{2} \cdot \lim_{R\to\infty} \ln\left|\frac{x-1}{x+1}\right| \Bigg|_4^R = \frac{1}{2} \cdot \lim_{R\to\infty} \left(\ln \underbrace{\left|\frac{R-1}{R+1}\right|}_{\xrightarrow[R\to\infty]{} 1} - \ln\left(\frac{3}{5}\right)\right) = \frac{1}{2} \cdot \left(\ln(1) - \ln\left(\frac{3}{5}\right)\right) = \frac{-1}{2} \cdot \ln\left(\frac{3}{5}\right).$$

Since $\int_4^\infty \frac{dx}{x^2-1}$ converges, $\sum_{n=4}^\infty \frac{1}{n^2-1}$ also converges.