Problem 11.5.58.: $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\frac{1}{2^5}+\cdots$

General term: $a_n = \frac{1}{2^{2n-1}} + \frac{1}{3^{n+1}}$ unfortunately does not work since $a_2 = \frac{1}{2^3}+\frac{1}{3^3}$ but $\frac{1}{3^3}$ is not part of the series. $a_n = \frac{1}{2^{2n-1}}+\frac{1}{3^{2n}}$ also works. ⚠

However: $a_n = \begin{cases} \frac{1}{2^n} & n \text{ odd.} \\ \frac{1}{3^n} & n \text{ even.} \end{cases}$ is a valid general term.

Compute the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2^{2(n+1)-1}}+\frac{1}{3^{2(n+1)}}}{\frac{1}{2^{2n-1}}+\frac{1}{3^{2n}}} = \frac{\frac{3^{2(n+1)}+2^{2(n+1)-1}}{2^{2(n+1)-1}\cdot 3^{2(n+1)}}}{\frac{3^{2n}+2^{2n-1}}{2^{2n-1}\cdot 3^{2n}}} = \frac{3^{2(n+1)}+2^{2(n+1)-1}}{3^{2n}+2^{2n-1}}\cdot\frac{2^{2n-1}\cdot 3^{2n}}{2^{2(n+1)-1}\cdot 3^{2(n+1)}} =$$

$$= \frac{3^{2(n+1)}+2^{2(n+1)-1}}{3^{2n}+2^{2n-1}}\cdot\frac{1}{4}\cdot\frac{1}{9}$$

No!

⚠ Careful! This is like putting parenthesis in: $1-1+1-1+\cdots = (1-1)+(1-1)+\cdots = 0+0+\cdots$

Compute the ratio test: if n is even then $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{3^n}} = \frac{3^n}{2^{n+1}} = \frac{1}{2}\cdot\left(\frac{3}{2}\right)^n$, if n is odd then:

$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{3^{n+1}}}{\frac{1}{2^n}} = \frac{2^n}{3^{n+1}} = \frac{1}{3}\cdot\left(\frac{2}{3}\right)^n$. Now: $\lim_{n\to\infty}\frac{1}{2}\cdot\left(\frac{3}{2}\right)^n = \infty$ and $\lim_{n\to\infty}\frac{1}{3}\cdot\left(\frac{2}{3}\right)^n = 0$, so the sequence $\frac{a_{n+1}}{a_n}$ does not have a limit, so the ratio test is inconclusive.

However since $3>2$ then $\frac{1}{3^n}\le\frac{1}{2^n}$ so $0\le a_n\le\frac{1}{2^n}$ for all n. Then: $\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty}\frac{1}{2^n}$ which is a converging geometric series with $r=\frac{1}{2}<1$. Then by the comparison test, $\sum_{n=1}^{\infty} a_n$ converges.

Problem 11.3.30.: $\sum_{n=1}^{\infty}\frac{n!}{n^3}$.

Note that $n! = n\cdot(n-1)(n-2)\cdot(n-3)!$ We want to simplify: $\frac{n!}{n^3} = \frac{n\cdot(n-1)(n-2)(n-3)!}{n\cdot n\cdot n}$ into something smaller. To decompose $n! = n\cdot(n-1)(n-2)(n-3)!$ we need $n \geq 4$. For such n we have:

$(n-1)\cdot(n-2) \geq n$. Then:

$$\sum_{n=1}^{\infty} \frac{n!}{n^3} = 1 + \frac{1}{4} + \frac{2}{3} + \sum_{n=4}^{\infty} \frac{n!}{n^3} = 1 + \frac{1}{4} + \frac{2}{3} + \sum_{n=4}^{\infty} \frac{n\cdot(n-1)(n-2)(n-3)!}{n^3} \geq 1 + \frac{1}{4} + \frac{2}{3} + \sum_{n=4}^{\infty} \frac{n\cdot n\cdot(n-3)!}{n^3} =$$

$$= 1 + \frac{1}{4} + \frac{2}{3} + \sum_{n=4}^{\infty} \frac{(n-3)!}{n} \underset{(n-3)!\,\geq\,1}{\geq} 1 + \frac{1}{4} + \frac{2}{3} + \sum_{n=4}^{\infty} \frac{1}{n},$$ which is a diverging harmonic series.

Problem 11.4.44.: The limit comparison test is valid for positive series: $0 \leq a_n \leq b_n$. In this case, $\sum a_n$ converges if and only if $\sum b_n$ converges, whenever $L = \lim_{n\to\infty} \frac{b_n}{a_n}$ is a positive value.

Use $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\cdot\left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$ to prove that the limit comparison test does not work for non-positive series. For this, we need $L = \lim_{n\to\infty} \dfrac{\frac{(-1)^n}{\sqrt{n}}\cdot\left(1+\frac{(-1)^n}{\sqrt{n}}\right)}{\frac{(-1)^n}{\sqrt{n}}}$ to be a finite value, and that one of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ or $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\cdot\left(1+\frac{(-1)^n}{\sqrt{n}}\right)$ converges, while the other diverges.

Note: $$L = \lim_{n\to\infty} \frac{\frac{(-1)^n}{\sqrt{n}}\cdot\left(1+\frac{(-1)^n}{\sqrt{n}}\right)}{\frac{(-1)^n}{\sqrt{n}}} = \lim_{n\to\infty}\left(1+\frac{(-1)^n}{\sqrt{n}}\right) = 1.$$

Also: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is an alternating series that converges by the Leibniz test.

Since $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges, the limit comparison test would say that $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\left(1+\frac{(-1)^n}{\sqrt{n}}\right)$ also converges.

Moreover, recall that if $\sum a_n$ converges and $\sum b_n$ converges, then $\sum (a_n - b_n) = \sum a_n - \sum b_n$ also converges. Now:

$$\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\left(1+\frac{(-1)^n}{\sqrt{n}}\right)} - \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}} \underset{\text{if and only if both converge}}{=} \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n} - \frac{(-1)^n}{\sqrt{n}}\right) = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}},$$ which does not converge.

both convergent　　　　　　　　　　divergent

Namely, we found two series $\sum a_n$, $\sum b_n$, such that $L = \lim\limits_{n\to\infty} \frac{b_n}{a_n} = 1$ and $\sum a_n$ converges, but $\sum b_n$ does <u>not</u> converge. Thus the limit comparison test is <u>not</u> valid for non-positive series.

<u>Problem 11.5.59.:</u> $\sum\limits_{n=1}^{\infty} \frac{c^n \cdot n!}{n^n}$.

a) By the ratio test: $\rho = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \cdots = \lim\limits_{n\to\infty} |c| \cdot \underbrace{\left(1+\frac{1}{n}\right)^{-n}}_{e^{-1}} = |c| \cdot e^{-1}$, so the series converges absolutely for $|c| < e$, and diverges for $|c| > e$.

$\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

$e^{\ln\left(\left(1+\frac{1}{n}\right)^n\right)} = e^{n \cdot \ln\left(1+\frac{1}{n}\right)}$

b) $\sqrt{2\pi} \approx 2.506628\ldots$

c) We want to use part b): $\lim\limits_{n\to\infty} \frac{e^n \cdot n!}{n^{1+\frac{1}{2}}} = \sqrt{2\pi}$. We compare with some series $\sum b_n$, which will diverge, and with $\lim\limits_{n\to\infty} \frac{\frac{e^n \cdot n!}{n}}{b_n} = L$ is finite. (We have $c = e$ so we are checking $\sum\limits_{n=1}^{\infty} \frac{e^n \cdot n!}{n}$)

By taking $b_n = \sqrt{n}$, now: $\lim\limits_{n\to\infty} \frac{\frac{e^n \cdot n!}{n}}{b_n} = \lim\limits_{n\to\infty} \frac{\frac{e^n \cdot n!}{n}}{\sqrt{n}} = \lim\limits_{n\to\infty} \frac{e^n \cdot n!}{n^{1+\frac{1}{2}}} = \sqrt{2\pi}$, so since $\sum\limits_{n=1}^{\infty} \sqrt{n}$ diverges then by the limit comparison test $\sum\limits_{n=1}^{\infty} \frac{e^n \cdot n!}{n}$ also diverges.