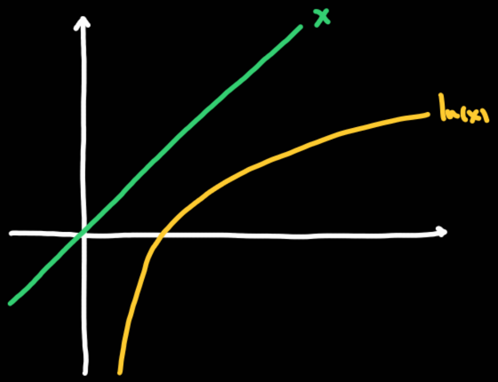
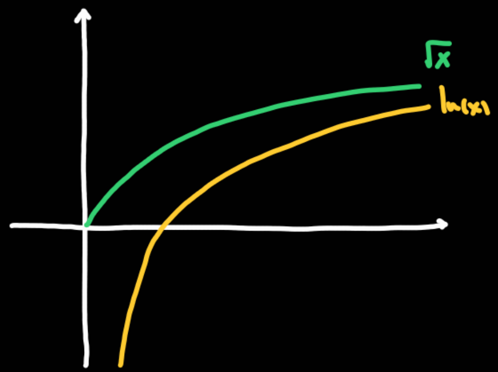
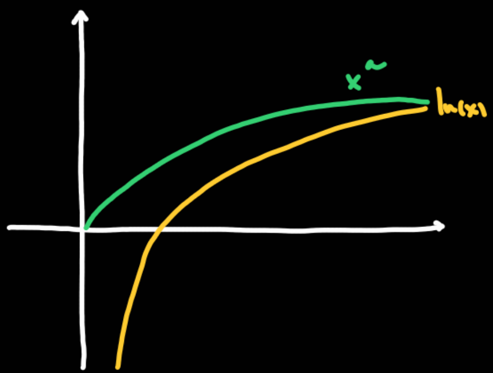


Note that $x > \ln(x)$



Note that $\sqrt{x} > \ln(x)$



We can find x^a for some real value a

such that $x^a < \ln(x)$ for some x.

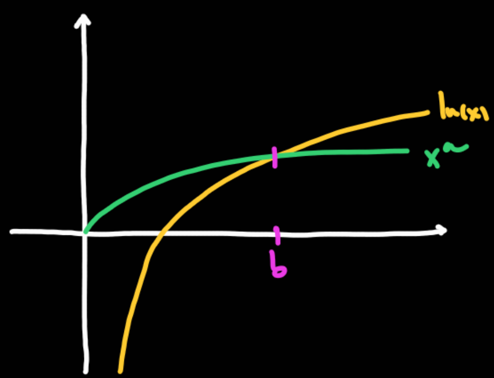
The "first" a with $x^a < \ln(x)$ for some x

must have finite x.

We have $x^a < \ln(x)$ for $x > b$,

$x^a > \ln(x)$ for $x < b$,

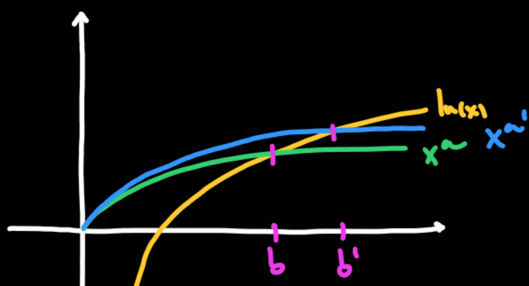
$x^a = \ln(x)$ for $x = b$.



By choosing a' slightly bigger than a,

we have $b' > b$.

"We are solving $x^a = \ln(x)$ for a".



Power series:

"Power series are infinite polynomials".

$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n \cdot x^n.$ $\qquad a_0, a_1, a_2, \ldots, a_n$ are real numbers.

$$T(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n \cdot x^n + \cdots = \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$T(x) = \sum_{n=0}^{\infty} a_n \cdot (x-c)^n.$$

The power series have radius of convergence R, which can be zero, some positive real number, or infinity. Inside the radius of convergence, power series are just long polynomials.

Proof on p.577.:

$$0 < S_2 < S_4 < \cdots < \lim_{N\to\infty} S_{2N} \leq \lim_{N\to\infty} S_{2N+1} < \cdots < S_3 < S_1$$

Since S_{2N} is strictly increasing and bounded above, so it has some limit $L = \lim_{N\to\infty} S_{2N}$.

Since S_{2N+1} is strictly decreasing and bounded below, so it has some limit $T = \lim_{N\to\infty} S_{2N+1}$.

Now:

$$T - L = \lim_{N\to\infty} S_{2N+1} - \lim_{N\to\infty} S_{2N} = \lim_{N\to\infty} (S_{2N+1} - S_{2N}) = \lim_{N\to\infty} a_{2N+1} = 0$$

So $T - L = 0$ so $T = L$.

Also: $\lim_{N\to\infty} S_{2N} \leq \lim_{N\to\infty} S_N \leq \lim_{N\to\infty} S_{2N+1}$ so $L = \lim_{N\to\infty} S_N = T$.

Problem 11.3.58: Determine convergence or divergence of $\sum_{n=2}^{\infty} \frac{(\ln(n))}{n^{9/8}}$.

Note that $\lim_{n\to\infty} \frac{\ln(n)}{x^a} = 0$ for all $a>0$ by l'Hôpital's rule. Hence x^a grows much faster than $\ln(n)$. So we can choose some $N>0$ such that for $n>N$ then $\ln(x)^{12} < x^{\frac{1}{16}}$, namely $\ln(x) < x^{\frac{1}{192}}$. Using this, we can prove convergence. (See Office Hours for November 15).

We want to compare $\sum_{n=2}^{\infty} \frac{\ln(n)^{12}}{n^{9/8}}$ with $\sum_{n=2}^{\infty} \frac{1}{n^p}$.

We will use that $\ln(n) < n^q$, so $\ln(n)^{12} < n^{12\cdot q}$. To make our life easier we can take $q = \frac{1}{12\cdot r}$, so $\ln(n)^{12} < n^{12\cdot q} = n^{\frac{1}{r}}$.

We will then divide:

$$\frac{\ln(x)^{12}}{n^{9/8}} < \frac{n^{12\cdot q}}{n^{9/8}} = \frac{n^{\frac{1}{r}}}{n^{9/8}} = \frac{1}{n^{\frac{9}{8}-\frac{1}{r}}} = \frac{1}{n^{\frac{18}{16}-\frac{1}{16}}} = \frac{1}{n^{17/16}}.$$ So $p = \frac{17}{16} > 1$.

$r=16$

How to prove convergence:

Ratio test.

Root test.

Comparison and Limit comparison. Useful: p-series, geometric series, harmonic series, ...

Integral test.

Divergence test.

Example 11.7.8.: $J = \int_0 \sin(x^2)\,dx$ is found by integrating term by term.

$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}$, substitute x^2 for x : $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{4n+2}}{(2n+1)!}$