Math 33A
Linear Algebra and Applications

Discussion 10

Problem 1.

Consider an $n \times m$ matrix A with $\text{rank}(A) = m$, and a singular value decomposition $A = U\Sigma V^T$. Show that the least-squares solution of a linear system $A\vec{x} = \vec{b}$ can be written as

$$\vec{x}^* = \frac{\vec{b} \cdot \vec{u_1}}{\sigma_1}\vec{v_1} + \cdots + \frac{\vec{b} \cdot \vec{u_m}}{\sigma_m}\vec{v_m}.$$

Solution: For some vector $\vec{x}^*$ to be a least-squares solution it just needs to satisfy $A\vec{x}^* = \text{proj}_{\text{im}(A)}(\vec{b})$. Since $\vec{u_1}, \ldots, \vec{u_n}$ are an orthonormal basis of $\mathbb{R}^n$ then

$$\begin{aligned} A\vec{x}^* &= A\left(\frac{\vec{b} \cdot \vec{u_1}}{\sigma_1}\vec{v_1} + \cdots + \frac{\vec{b} \cdot \vec{u_m}}{\sigma_m}\vec{v_m}\right) = \vec{b} \cdot \vec{u_1}\frac{A\vec{v_1}}{\sigma_1} + \cdots + \vec{b} \cdot \vec{u_m}\frac{A\vec{v_m}}{\sigma_m} \\ &= (\vec{b} \cdot \vec{u_1})\vec{u_1} + \cdots + (\vec{b} \cdot \vec{u_m})\vec{u_m} = \text{proj}_{\text{im}(A)}(\vec{b}) \end{aligned}$$

because $\vec{u_1}, \ldots, \vec{u_m}$ is an orthonormal basis of $\text{im}(A)$. Thus $\vec{x}^*$ is a least-squares solution, as desired.

Problem 2($\star$).

Consider the 4×2 matrix

$$A = \frac{1}{10}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}.$$

Find the least-squares solution of the linear system

$$A\vec{x} = \vec{b} \quad \text{where} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solution: We can read off the decomposition of A the following

$$\vec{u_1} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u_2} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \vec{v_1} = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}, \vec{v_2} = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}, \sigma_1 = 2, \sigma_2 = 1$$

so by the above we find

$$\vec{x}^* = \frac{\vec{b} \cdot \vec{u_1}}{\sigma_1}\vec{v_1} + \frac{\vec{b} \cdot \vec{u_2}}{\sigma_2}\vec{v_2} = \begin{bmatrix} -1/10 \\ -16/5 \end{bmatrix}.$$

Problem 3.

(a) Explain how any square matrix A can be written as $A = QS$, where Q is orthogonal and S is symmetric positive semidefinite. This is called the polar decomposition of A.

(b) Is it possible to write $A = S_1 Q_1$, where Q_1 is orthogonal and S_1 is symmetric positive semidefinite?

Solution:

(a) Let $A = U\Sigma V^T$ be the singular value decomposition of A. Set $Q = UV^T$ and $S = V\Sigma V^T$, we can rewrite

$$A = U\Sigma V^T = UV^T V\Sigma V^T = QS$$

where Q is orthogonal because it is the product of orthogonal matrices, and S is symmetric since

$$S^T = (V\Sigma V^T)^T = (V^T)^T \Sigma^T V^T = V\Sigma V^T$$

because Σ only has non zero entries in its diagonal. Moreover, since S is similar to Σ then they have the same eigenvalues, and the eigenvalues of Σ are its diagonal entries, which are all positive or zero. Thus S is positive semidefinite.

(b) Yes. Set $S_1 = U\Sigma U^T$ and $Q_1 = UV^T$ and rewrite

$$A = U\Sigma V^T = U\Sigma U^T UV^T = S_1 Q_1$$

where, as we just saw, Q_1 and S_1 are orthogonal and symmetric positive semidefinite.

Problem 4.

Find a polar decomposition $A = QS$ for

$$A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}.$$

Draw a sketch showing $S(C)$ and $A(C) = Q(S(C))$, where C is the unit circle centered at the origin.

Solution: We compute its singular value decomposition and obtain

$$A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix} = \left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\right)\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right) = U\Sigma V^T$$

so

$$Q = \left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix},$$

$$S = \left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\right)\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix},$$

and

$$A = \left(\frac{1}{5}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}\right)\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}.$$

Problem 5.

Show that a singular value decomposition $A = U\Sigma V^T$ can be written as

$$A = \sigma_1 \vec{u_1}\vec{v_1}^T + \cdots + \sigma_r \vec{u_r}\vec{v_r}^T.$$

Solution: We can rewrite the singular value decomposition of A as

$$A = U\Sigma V^T = \begin{bmatrix} | & & | \\ \vec{u_1} & \cdots & \vec{u_n} \\ | & & | \end{bmatrix}\begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}\begin{bmatrix} — & \vec{v_1} & — \\ & \vdots & \\ — & \vec{v_m} & — \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ \vec{u_1} & \cdots & \vec{u_n} \\ | & & | \end{bmatrix}\begin{bmatrix} — & \sigma_1\vec{v_1} & — \\ & \vdots & \\ — & \sigma_r\vec{v_r} & — \\ & 0 & \\ & \vdots & \\ & 0 & \end{bmatrix}$$

$$= \sigma_1 \begin{bmatrix} | \\ \vec{u_1} \\ | \end{bmatrix}\begin{bmatrix} — & \vec{v_1} & — \end{bmatrix} + \cdots + \sigma_1 \begin{bmatrix} | \\ \vec{u_1} \\ | \end{bmatrix}\begin{bmatrix} — & \vec{v_1} & — \end{bmatrix} = \sigma_1 \vec{u_1}\vec{v_1}^T + \cdots + \sigma_r \vec{u_r}\vec{v_r}^T$$

giving the desired decomposition.

Problem 6.

Find a decomposition $A = \sigma_1 \vec{u_1}\vec{v_1}^T + \sigma_2 \vec{u_2}\vec{v_2}^T$ for

$$A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}.$$

Solution: We compute its singular value decomposition and obtain

$$A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix} = \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \right) \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \right) = U\Sigma V^T.$$

We can read off this decomposition the following

$$\vec{u_1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \vec{u_2} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v_1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{v_2} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \sigma_1 = 10, \sigma_2 = 5$$

so

$$\begin{aligned} A &= 10 \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \end{bmatrix} \right) + 10 \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \end{bmatrix} \right) \\ &= \begin{bmatrix} 8 & -2 \\ -8 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}. \end{aligned}$$