Math 33A
Linear Algebra and Applications

Discussion 6

Problem 1($\star$).

The following is one way to define the quaternions, discovered in 1843 by the Irish mathematician Sir W. R. Hamilton. Consider the set H of all 4×4 matrices M of the form

$$M = \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix}$$

where p, q, r, s are arbitrary real numbers. We can write M more succinctly in partitioned form as

$$M = \begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix}$$

where A and B are rotation–scaling matrices.

(a) Show that H is closed under addition: If M and N are in H, then so is $M + N$.

(b) Show that H is closed under scalar multiplication: If M is in H and k is an arbitrary scalar, then kM is in H.

(c) The above show that H is a subspace of the linear space $\mathbb{R}^{4\times 4}$. Find a basis of H, and thus determine the dimension of H.

(d) Show that H is closed under multiplication: If M and N are in H, then so is MN.

(e) Show that if M is in H, then so is M^T.

(f) For a matrix M in H, compute $M^T M$.

(g) Which matrices M in H are invertible? If a matrix M in H is invertible, is M^{-1} necessarily in H as well?

(h) If M and N are in H, does the equation $MN = NM$ always hold?

Problem 2.

Consider a consistent system $A\vec{x} = \vec{b}$.

(a) Show that this system has a solution $\vec{x_0}$ in $(\ker A)^{\perp}$. Justify why an arbitrary solution $\vec{x}$ of the system can be written as $\vec{x} = \vec{x_h} + \vec{x_0}$, where $\vec{x_h}$ is in $\ker(A)$ and $\vec{x_0}$ is in $(\ker A)^{\perp}$.

(b) Show that the system $A\vec{x} = \vec{b}$ has only one solution in $(\ker A)^{\perp}$.

(c) If $\vec{x_0}$ is the solution in $(\ker A)^{\perp}$ and $\vec{x_1}$ is another solution of the system $A\vec{x} = \vec{b}$, show that $||\vec{x_0}|| < ||\vec{x_1}||$. The vector $\vec{x_0}$ is called the minimal solution of the linear system $A\vec{x} = \vec{b}$.

Problem 3.

Define the term minimal least-squares solution of a linear system. Explain why the minimal least-squares solution $\vec{x}^*$ of a linear system $A\vec{x} = \vec{b}$ is in $(\ker A)^{\perp}$.