Math 33A
Linear Algebra and Applications

Midterm 1 — Lecture 1

Instructions: You have 50 minutes to complete this exam. There are 6 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. Please write your solutions in the space provided below the statement of the problem, box your final answer, show all your work legibly, and clearly reference any results that you use. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Once the 50 minutes have elapsed, you are not allowed to continue writing and you are not allowed to communicate with anybody except the administrators of the exam. Please follow their requests at all times. Failure to comply with any of these instructions may have repercussions in your final grade.

Name: ______________________________

ID number: ___________________________

Section: _____________________________

Question	Points	Score
1	10	
2	18	
3	18	
4	18	
5	18	
6	18	
Total:	100	

Problem 1. *10pts.*

Determine whether the following statements are true or false. If the statement is true, write **T** over the line provided before the statement. If the statement is false, write **F** over the line provided before the statement. Do NOT write "true" or "false".

(a) __**F**__ If matrix A is in reduced row-echelon form, then at least one of the entries in each column must be 1.

(b) __**T**__ Let A be an $n \times m$ matrix. Then you can transform $\text{rref}(A)$ into A by a sequence of elementary row operations.

(c) __**T**__ The formula $(A^5)^{-1} = (A^{-1})^5$ holds for all invertible matrices A.

(d) __**F**__ If $A^2 = I_2$, then matrix A must be either I_2 or $-I_2$.

(e) __**T**__ If A and B are $n \times n$ matrices, and vector $\vec{v}$ is in the kernel of both A and B, then $\vec{v}$ must be in the kernel of matrix AB as well.

Additional page for solutions 1

Problem 2. *18pts.*

Find all solutions of the linear system

$$\begin{aligned} x + 2y + 3z &= a \\ x + 3y + 8z &= b \\ x + 2y + 2z &= c \end{aligned}$$

where a, b, c are arbitrary constants.

Solution: The solutions are $x = 10a - 2b - 7c$, $y = -6a + b + 5c$, $z = a - c$.

Additional page for solutions 2

Problem 3. *18pts.*

Determine whether the vector

$$\begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$$

is a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 10 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 6 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ 2 \\ 3 \end{bmatrix}.$$

Solution: The last three vectors form a basis of $\mathbb{R}^3$, so the first vector is a linear combination of them.

Additional page for solutions 3

Problem 4. *18pts.*

Find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution: The inverse is

$$\begin{bmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix}.$$

Additional page for solutions 4

Problem 5. *18pts.*

Consider the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by the orthogonal projection onto the plane $x + 2y + 3z = 0$.

(a) Find a basis of the image.

(b) Find a basis of the kernel.

Hint: It is not necessary to find the matrix of the linear transformation.

Solution: The linear transformation has image

$$\text{im}(T) = \{(x, y, z) \in \mathbb{R}^3 | x + 2y + 3z = 0\},$$

namely the whole plane $x + 2y + 3z = 0$, with basis

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The linear transformation has kernel

$$\ker(T) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right),$$

namely the line spanned by the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

with basis

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

Additional page for solutions 5

Problem 6. *18pts.*

Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that

$$\begin{bmatrix}1\\2\end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix}3\\5\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}3\\4\end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix}2\\3\end{bmatrix}.$$

Solution: We are looking for a matrix S such that $S\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$ and $S\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$. This has solution $S = \begin{bmatrix}12 & -7\\14 & -8\end{bmatrix}$, so the desired basis is $\left\{\begin{bmatrix}12\\14\end{bmatrix}, \begin{bmatrix}-7\\-8\end{bmatrix}\right\}$.

Additional page for solutions 6